DRAINAGE TO PARALLEL DRAINS

by

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DRAINAGE TO PARALLEL DRAINS

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CERTIFICATE

This is to certify that the thesis entitled 'Drainage Studies to Parallel Drains' submitted by Sri Ajay Kumar in partial fulfilment of requirements for the degree of Master of Technology at Indian Institute of Technology Kanpur is a record of bonafide research work carried out by him under my supervision and guidance. The work embodied in this thesis has not been submitted elsewhere for a degree.

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AJAY KUMAR

DRAINAGE

TO

PARALLEL DRAINS

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Following is the list of some important symols and notations used in this thesis

b_2 Thickness of clogging layer at left boundary ,

b' Thickness of semi-previous layer separating two aquifers ,

c Total resistance to inter-aquifer flow ,

c_u Resistance to inter-aquifer flow due to unconfined aquifer ;

c_1 Resistance of clogging layer at left boundary ,

c_2 Resistance of clogging layer at right boundary ,

c' Resistance of thin semi-pervious separating layer ,

d Height of drains measured from datum ,

d_a Average depth of flow in unconfined aquifer ,

e Potential rate of evaporation ,

e_1 Constants used in defining transpiration rate ,

e_2 Constants used in defining transpiration rate ,

f Function of x corresponding to initial condition ,

h Verticle height to the pheratic surface measured from bed level;

h_m Maximum height of water table ,

h_o Verticle height to the artisen head line measured from datum;

h_r Verticle height to water level in right reservoir ,

h_u Piezometric head in unsaturated soil ,

h_y Verticle height to pheratic surface measured from datum ,

h_1 Verticle height to water level in left reservoir ,

i Rate of infiltration ,

I Slope of bed ,

k' Permeability of semi-pervious layer at the bottom ,

k_1 Permeability of clogging layer at left boundary ,

k_2 Permeability of clogging layer at right boundary ;

k_u Unsaturated permiability ,

k_s Saturated permeability ,

K Nondimensional unsaturated permeability,

K_v Transpiration efficiency ,

l_{c1} Distance from left boundary to a point where pheratic surface joins artesian head line ,

l_{c2} Distance from right boundary to a point where pheratic surface joins artesian head line ,

l_x . Length of flow domain in x-direction ;

l_z · Length of flow domain in z-direction ;

L_{rl} Distance between two reservoirs ;

p Slope of artisen head line ;

p_b . Bubbling pressure of soil matrix ,

P_u Soil matrix potential of the unsaturated soil ,

P . Non-dimensional soil matrix potential ;

P_u^* Threshold soil matrix potential for transpiration ,

q Artisen head at left end ,

q_x Flow rate in positive x-direction ,

q_z Flow rate in positive z-direction ,

r_p Resistance to moisture flow in plant ,

r_s Resistance to moisture flow in soil ,

S Non dimensional slope ,

t time ,

t_{d1} Time from begining of first rainfall to the time when drying starts ,

t_{r1} Duration of first rainfall ,

t_{s1} Time from begining of first rainfall till soil surface gets saturated ,

t_{r2} Time since the start of first rainfall to the time when second rainfall starts ,

T Nondimensional time ;

t_a Actual rate of transpiration ,

t_p Potential rate of transpiration ;

T_a Actual rate of transpiration per unit depth of root zone ,

T_p Potential rate of transpiration per unit depth of root zone ,

w Recharge from top ,

W Nondimensional recharge from top ,

x Horizontal distance measured from origin ,

X Non-dimensional horizontal distance ;

y Verticle distance measured from the datum ,

z Verticle distance measured from ground surface ,

Z . Non-dimensional vertical distance measured from ground surface ,

χ Index used in defining transpiration rate - effective saturation relationship ,

δ Diffusivity of unsaturated soil ,

η Effective porosity ,

ϕ Effective saturation ,

ϕ^* Effective saturation corresponding to threshold moisture content ,

ϕ_{wp} Effective saturation corresponding to wilting point ,

γ Specific weight of soil water ,

θ . Volumetric moisture content ,

θ_s Volumetric moisture content at saturation ;

λ Pore size distribution index ,

ρ Conductance of the semi-pervious layer ;

ω · A macroscopic physical property of soil which indicates the amount of work done per unit volume of soil required to drain a saturated soil to the wilting point ,

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CHAPTER 1

INTRODUCTION

1.1 INTRODUCTION

Increased demand of water supply for a variety of purposes require conjuctive use of surface and ground water resources Recently lot of emphasis has been put on ground water resources since the surface water resorces are limited and less dependable Ground water is available in a number of water bearing formations beneath the surface of earth called aquifers The topmost aquifer usually occurs with a pheratic surface and is called an unconfined aquifier Flow in unconfined aquifers is important not only from the point of view of extraction of water but also from agricultural point of view In fact it is this aquifer which is subjected to maximum fluctutations Some of the problems involving unconfined flow are

(i) Flow between a river and an unconfined aquifer with or without recharge to the aquifer ,

(ii) Flow through and below an earth dam ,

(iii) Flow to a well in an unconfined aquifer ;

(iv) Flow between two parallel drains or ditches provided for controlling ground water ,

(v) Flow in an unconfined aquifer overlying a semi-pervious strata ;

(vi) Artificial recharge problems

All the problems involve finding of water table heights as a function of space and time. Though man is concerned with uncon-

fined aquifers for a long time analytical solutions for only a few problems exist even to date in contrast with the confined aquifers This is mainly because of the following reasons

(a)The partial differential equation governing the ground water flow in unconfined aquifer is non-linear

(b)The boundary at the top of unconfined aquifer is not known until a solution is obtained,and the solution depends on the boundary whereas the flow domain in a confined aquifer is initially known

The exact analytical solutions for many complicated ground water flow problems are still not available owing to the above mentioned difficulties Numerical methods are usually employed in such cases. Analytical solutions, if available are always preffered, even if they are often based on simplifying assumptions Analytical solutions provide following advantages

(a) It is convenient to use analytical solution, if one exists.

(b) They are useful in verifying numerical methods

(c) The effect of various parameters can be studied in a quick and easy way with analytical solutions

(d) Also such solutions are useful in priliminary studies of aquifer developement schemes

Various approaches have been used by investigators to find analytical solutions Theory of equipotential flow has been used extensively for finding analytical solution of two dimensional problems. Conformal transformations are used to simplify the flow situation But this approach has serious limitations for analysing transient flow situations

The approximate analysis of shallow flow in unconfined aquifer over an impermeable bed was initiated by Dupuit by making the well known Dupuit's assumptions These assumptions reduce 2-D flow situation to 1-D and 3-D flow situation to 2-D

Boussinesq derived the governing equation for the unsteady state flow over sloping fully impervious bed with the help of Dupuit's assumptions Additional terms are required to be added to the original Boussinesq equation to take into account rainfall recharge and recharge from or leakage to the underlying aquifer

To overcome the difficulty imposed by nonlinearity, various types of linearisation have been proposed in the literature. The linearised Boussinesq equation provides a powerful tool for treating flow problems in unconfined aquifer in order to find an engineering solution Once the equations are linearsed the principle of superposition can be utilised Such linearisation is of much value for treating transient flow problems.

When the underlying strata is not fully impervious, leakage from or to the aquifer takes place in addition to recharge from top, if present. The magnitude of the inter-aquifer flow will depend upon the difference of heads in unconfined aquifer and underlying aquifer, and in general this will be nonuniform. The recharge from top however is usually taken to be uniform If the the piezometric surface in the underlying aquifer is too high, artesian condition may occur Usually such condition occurs in the following circumstances .

(a) In vally bottomlands where alluvial fans are often underlain by lensed semi-pervious formations

(b) In coastal plains where the high ocen level causes such problem Typical flow situation at a Dutch polder is of this type

(c) At foundation construction sites

(d) When land is protected by river levees and the artesian pressure is imposed by navigation improvements or periodically imposed by flood levels

1.2 OBJECTIVE

The purpose of this study is to investigate flow problems in unconfined aquifer on sloping semi-pervious beds Following types of problems have been considered in this context:

(1) An exact solution for the steady state flow over horizontal semipervious bed when artesian head line is also horizontal

(2) An exact solution for the unsteady state flow over horizontal barrier between parallel drains provided on the barrier

(3) A linearised solution for steady state flow over sloping bed when artesian head line is also parallel to the bed.

(4) A general solution for unsteady state flow for

(i) Transient recharge from top

(ii) Leakage from or to unconfined aquifer

(iii) Arbitrary initial condition

(iv) Different boundary conditions

(v) Horizontal as well as sloping bed

1.3 APPROACH

Methods of Laplace Transform and Fourier Transform have been extensively used for solving partial differential equations. Many investigators have used this approach to find valuable

solutions for ground water flow problems. But in the present study, method of substitution is used Simplified equation thus obtained can be easily solved for a wide variety of boundary conditions In case of transient flow problems the substitutions will make the boundary conditions transient In these problems Duhamel's theorem will be used to deduce solution for transient boundary conditions. The transformed initial condition can be easily incorporated using Fourier series Then a process of back substitution will lead to the desired results

The above mentioned approach has the advantage that only a simple partial differential equation requires to be solved for which solution exists in literature.Again this approach is also applicable to problems of other fields where similar partial differential equation is required to be solved e.g. solute transport problems

CHAPTER 2

LITERATJRE REVIEW

Literature connected with flow towards drains and streams is reviewed in a chronological order in the following section :

The analysis of flow in unconfined aquifer was initiated by Dupuit (1863) by making the assumptions that the flow lines are horizontal and hydraulic gradient at any section is equal to the slope of pheratic surface at any section These assumptions reduce a 2-D problem to 1-D and 3-D problem to 2-D

Using Dupuit's assumptions Boussinesq (1904) derived the partial differential equation governing unconfined ground water flow in non-deformable homogeneous porous media lying over a sloping impervious bed

Hooghoudt (1940) used Dupuit-Forchheimer approach to find an exact solution for steady state drainage of uniform rainfall by parallel drains in a horizontal aquifer To account for convergence he proposed the use of equivalent depth instead of actual height of drains above impervious barrier

Muskat (1946) found a solution for flow between two parallel drains over horizontal semi-pervious bed by conformal mapping He assumed that the horizontal line passing through the maximum height of the water table is a stream line, drain radius was negligibly small and flow near drains is essentially radial. Recharge from top or the presence of any semi-pervious layer at bottom was not considered.

Werner (1953, 1957) introduced recharge term in the Boussinesq equation and using transformation $h^2=y$, linearised the equation and solved it for flow between parallel reservoirs and for flow between ground water divide and a reservoir using Laplace transformation approach

Glover (1953,1954,1960) linearised the Boussinesq equation using average depth of flow and found analytical solution for the case of flow between parallel drains over impervious beds

Kirkham (1958) considered the problem of seepage of steady rainfall through soil into drains as a boundary value problem and applied the potential theory approach to find a solution.

Kraijenhoff van de leur (1958,1962) solved the Boussinesq equation for predicting ground water table between two parallel drains in a horizontal aquifer subjected to constant recharge.

Maasland (1959) was one of the first to treat the problem of transient recharge analytically He approximated the input terms by a piecewise constant function and used the solution for constant rate of recharge to construct the solution for transient recharge

Schmid and Luthin (1964) found implicit solution for steady state flow between parallel drains over sloping barrier using method of substitutions Spacing of drains was found explicitly The maximum height was compared with the experimental solution obtained with the help of Hele-Shaw model and was found to be sufficiently accurate

Moody (1966) used numerical approach to provide correct solution in graphical form for flow between two parallel drains over a horizontal impervious barrier The convergece to drains was accounted for by using Hooghoudt's equavalent depth for

which he developed simpler expressions The error due to use of simpler expressions was found to be less than 7%

Hinesly and Kirkham (1966) considered the flow between two parallel drains over horizoi tal semi-pervious bed subjected to a steady recharge as a boundary value problem which they solved with the help of Fourier Transform The head loss of water flowing above the drain level was neglected and drains were assumed to run half full The maximum height of water table was taken as the head midway between the drains

Luthin and Guitjens (1967) studied the transient water table falling between parallel drains in a sloping aquifer with the help of a Hele-Shaw model They gave empirical relations to estimate drain spacing and indicated that rate of fall of water table was almost independent of slope for slopes upto 30°

Chauhan et al. (1968) obtained an analytical solution for time varying end conditions for flow in a sloping aquifer. They also obtained experimental solution wth a Hele-Shaw model and found that the Boussinesq equation is capable of characterising transient water table for drains penetrating to the full depth Of impermeable layer upto 8% slopes

E C Childs (1971) pointed out that for flow over sloping beds largely governed by bed slopes, streamlines will be parallel to bed instead of being horizontal,the equipotential line being normal to it.Each equipotential line is still labelled by the height of point where it cuts the pheratic surface measured from a horizontal datum Thus he obtained a new expression for flow using Darcy's law and found an exact analytical solution for flow between two parallel drains transverse to the bedslope

for the case of no recharge

Marino (1973) considered a stream aquifer system over an impermeable base and derived analytical solutions for the cases when bank of stream was clogged due to the presence of a thin layer for both semi-infinite and finite aquifers The initial water table was taken to be horizontal and in case of finite aquifer the other boundary was taken to be of no flux type No recharge from top was considered and Laplace Transform was used to obtain a solution He (1974 a,b,c) also presented analytical solutions for transient pheratic surface in semi-infinte and finite horizontal unconfined aquifers which recieves localised recharge at a constant rate and discharges into a neighbouring reservoir The initial water table was taken to be horizontal and Laplace Transformation was applied to find a solution

G.D Towner (1975) employed proper substitutions in Childs' (1971) differential equation for steady state flow between two parallel ditches over sloping bed with uniform rainfall,thereby obtaining similar integrations as Schmid and Luthin Thus implicit analytical results were found for flows controlled largely by sloping bed

Singh and Jacob (1977) used the method of functional transformation for linearisation of Boussinesq equation.The equation so linearised was solved for flow between parallel drains or streams with unsteady recharge for arbitrary initial conditions and transient boundary condition of Dirichlet type. Excellent agreement was found between linearised and numerical solutions

Mualem and Bear (1978) proposed a new transformation and linearisation technique for steady pheratic flow over a sloping semi-pevious bed. The solution of the linearised equation was

verified with a Hele-Shaw analog Excellent agreement was found between analytical and experimental results

T G Chapman (1980) gave a simpler expression for discharge over a sloping impervious bed utilizing a new approximation in Childs'(1971) expression The new approximation was found valid upto a slope of 30^{o}

Sakkas et al (1981) fitted simple equations for Hooghoudt equavalent depth by using method of least squares and indicated that maximum errors were less than 1%

Rai and Singh (1981) obtained a solution for transient water table in a horizintal semi-infinite aquifer subjected to localised transient recharge linearly decreasing with time and then remaining constant using Laplace Transformation approach.

Bruggeman (1983) considered the steady flow between parallel drains over semi-pervious bed as a boundary value problem as Hinsley and Kirkham (1966) did, but some extra parameters such as the resistance of a thin layer at bottom, resistance against entry of water into drains and anisotropy were taken care of

Wesseling and Wesseling (1984) used the Dupuit-Forchhemer approach to find the solution for flow between parallel drains over horizontal semi-pervious bed subjected to steady recharge The resistance of a thin loamy layer at bottom was taken care of Using Glover's type of linearisation they found a solution for steady as well as unsteady flow

Tolikas et al. (1984) presented an approximate analytical solution for the problem of recharge of a semi-infinite aquifer from a stream with uniform initial condition and a step increase of piezometric head on the boundary. The 1-D problem was reduced to an ordinary differential equation through Boltzman

Transformation and a technique exploiting some basic characteristics of the exact solution leads to an approximate polynomial solution of the problem

Yates and Warrick (1985) developed an exact solution for steady state flow in a sloping Dupuit aquifer subjected to a constant uniform recharge In case of leaky substratum they suggest that aritesian recharge should be added to the recharge from top assuming it to be uniform

Latinopoulos (1986) developed an analytical solution for pheratic surface in finite horizontal aquifers when one of the boundary condition was of third type, the other being either of Dirichlet type or Newmann type. The initial condition was taken as horizontal water table and a periodic recharge (constant or zero) was considered to take place The solution was obtained using Laplace and Finite Fourier Transforms.

Bazaraa et al. (1986) studied the aritesian and anisotripic effects on drain spacing. They indicated that large piezometric heads may necessiate small spacings which may be uneconomical The anisotropic effects were studied by converting ansiotropic system into a fictious isotropic system keeping the vertical dimensions unchanged It was found that drain spacing increases as anisotropy (K_{sh}/K_{sv}) increases for impermeable beds For permeable beds and shallow depth of aquifers spacing increases initially with anisotropy till magnitude of aritesian flux is reduced a level comparable to rainfall recharge. Thenafter it decreases.

Wolters (1986) compared five different formulae given by Muskat (1946),Young (1986),Hinesly (1966),Bruggeman (1983) and Wesseling and Wesseling (1984) for certain cases He indicated

that Wesseling and Wesseling (1984) formula in combination with Hooghoudt equavalent depth gives equivalent results as Hinesly and Bruggeman, and maximum height of water table so obtained is higher than those given by Muskat's and Young's formulae

Youngs (1986) used the concept of potential flow theory and conformal mapping to find out water table heights between two partially penetrating drains draining the artesian water in a horizontal aquifer The solution involves integrals which can be evaluated only by numerical methods.To avoid this difficulty he gave an empirical equation which fitted the correct solution well

Ram and Chauhan (1987) found analytical solutions for rise of water table between two parallel drains in response to steady, linearly increasing and exponentially decreasing recharges with zero initial condition The analytical solutions were verified experimentally with a Hele-Shaw model for slopes varying from 0-0 078 % Large errors were found near ends due to convergence of flow The analytical solution was found acceptable though the need of proper selection of average depth of flow in this case (when drains are on the barrier) was indicated.

CHAPTER 3

FORMULATION OF THE PROBLEM

3.1 INTRODUCTION

Generally ground water flow problems in unconfined aquifers involve the determination of phreatic surface and its variation with respect to space and time Sometimes the rate of leakage to drains and streams is also desired Some of the common flow situations are flow between two parallel channels (*fig*-3 1a), flow between a no flux boundary and a river (*fig*-3.1b) and flow between two parellel pipe drains (*fig*-3 1c) employed to control the water table The bed of the aquifer may be sloping or semi-pervious or both (*fig*-3.1d,e). In case aquifer bed is sloping, the phreatic surface will no longer be symmetrical and hence location of ground water divide cannot be predetermined as in the case of aquifer with horizontal bed In all the above cases a transient recharge may be taking place from top In practice such a recharge arises due to rainfall or due to excess irrigation or artificially applied water at the surface In addition to the recharge from top, flow of water to (*fig*-3.1d) or from (*fig*-3 1e) the unconfined aquifer may take place if the aquifer bed is not fully impervious.Such situations are quite common in practice In a typical situation (*fig*-3 1d,e) most frequently encountered in practice, the unconfined aquifer is underlain by thin semi-pervious stratum which in turn overlies an aquifer If the piezometric surface in this semi-confined aquifer is higher than phreatic surface, recharge to unconfined aquifer will take

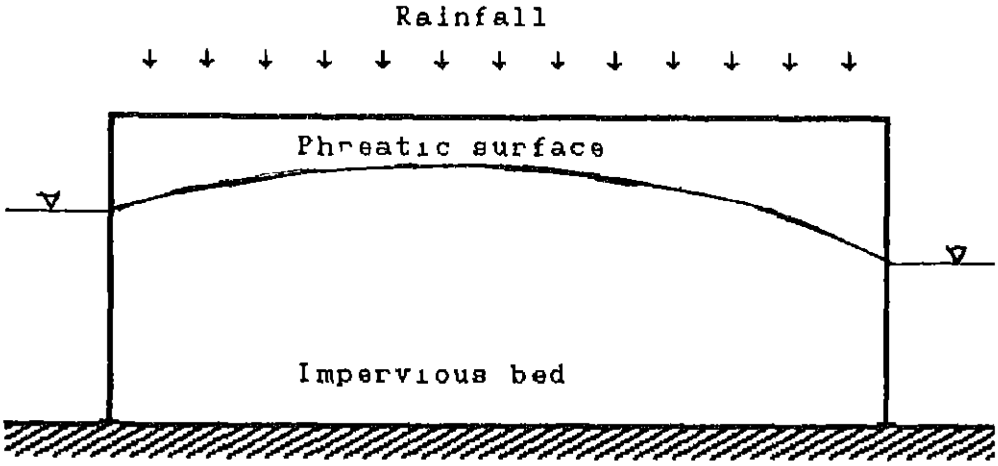


Fig. 3.1 (a)

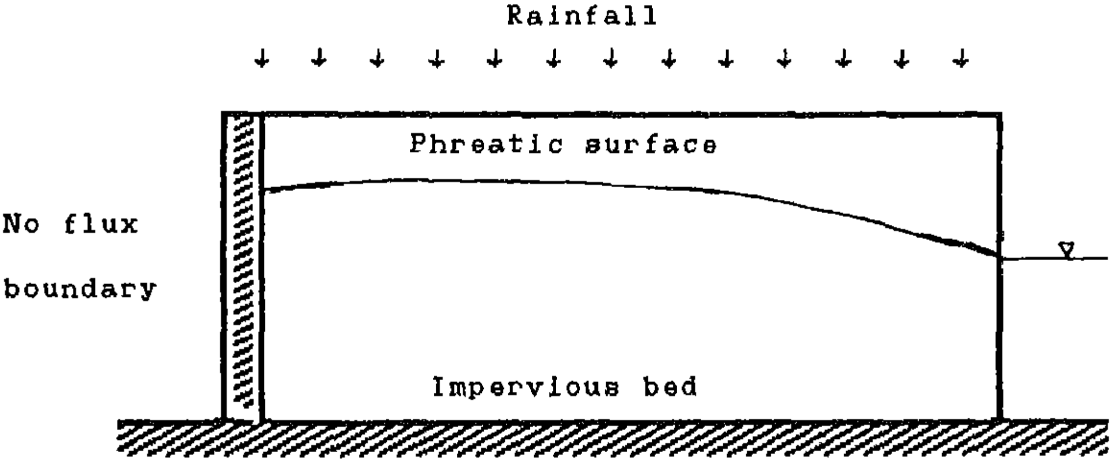


Fig. 3.1 (b)

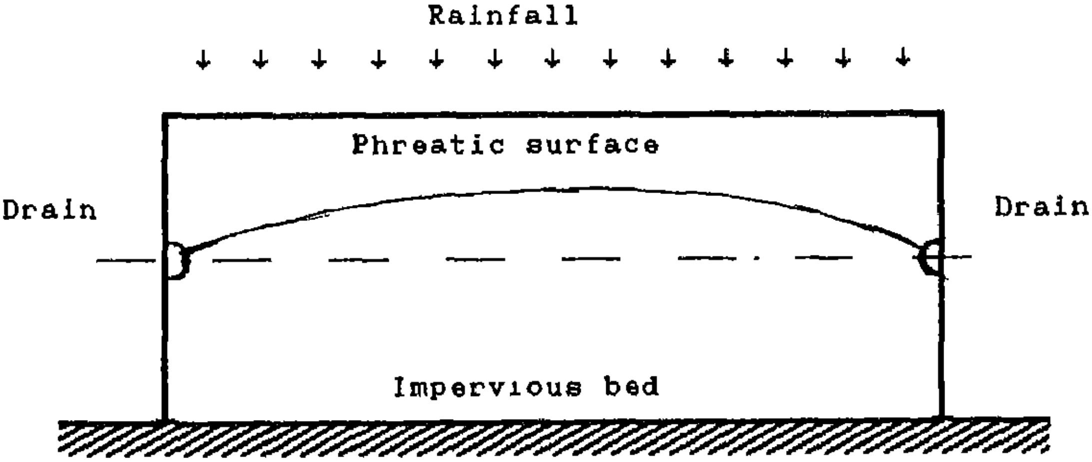


Fig. 3.1 (c)

Fig. 3.1 Some ground water flow situations

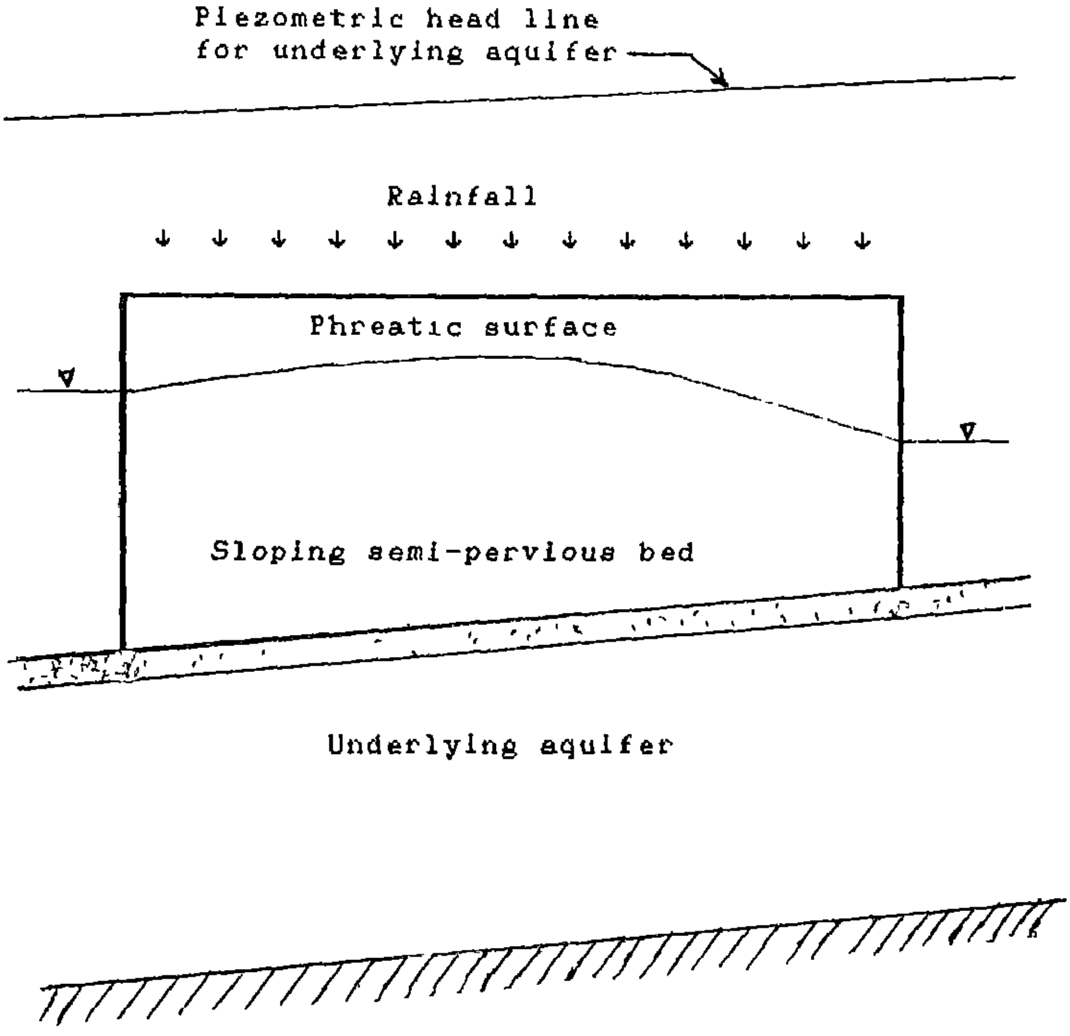


Fig 3 1 (d)

Fig. 3.1 (contd) Some ground water flow situations

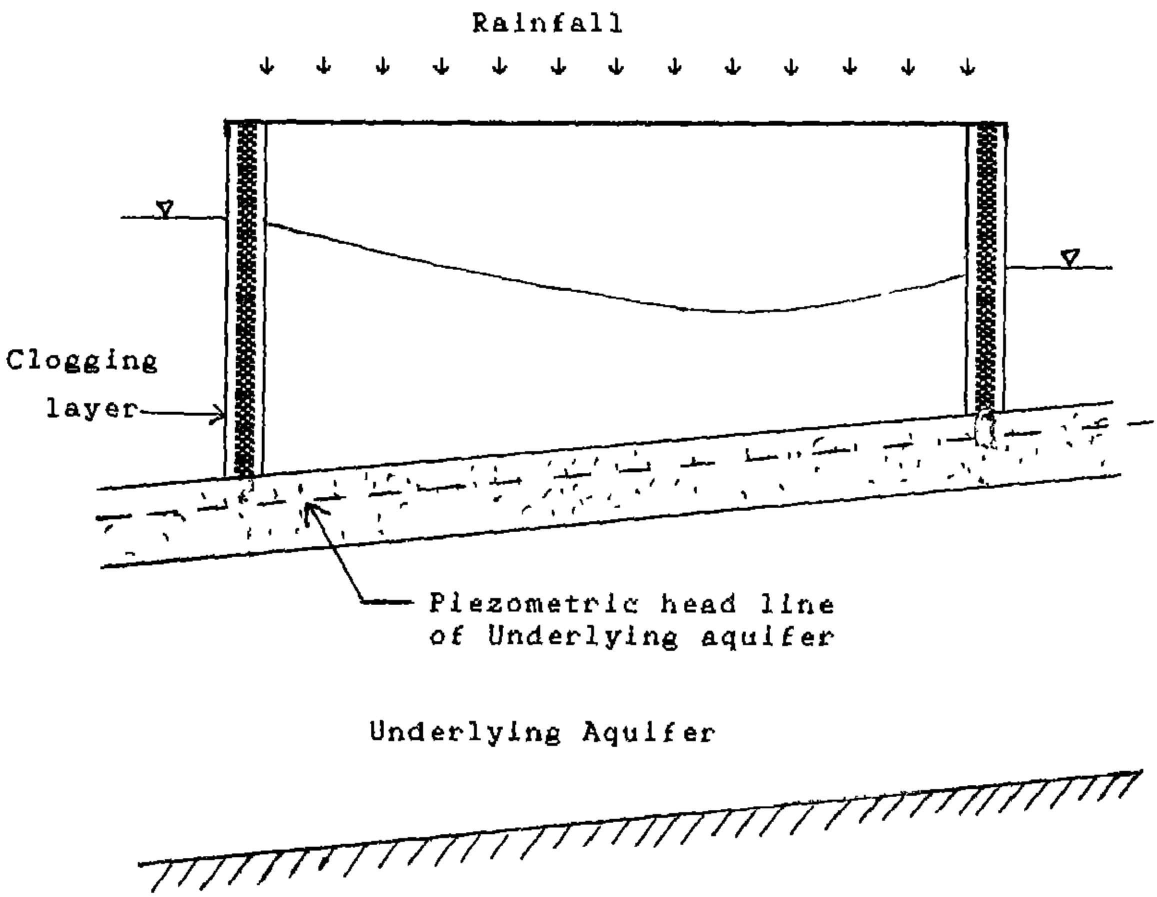


Fig. 3 1 (e)

Fig 3.1 (contd.) Some ground water flow situations

place This recharge may be appreaciable necesiating provision of drains to eliminate this unwanted water If the piezometric surface is low, the unconfined aquifer will be losing water The recharge to or from unconfined aquifer in magnitude will depend upon the difference between the elevations of piezometric and phreatic surface and since either may not be uniform,it will be nonuniform Therefore the practice of adding it to the recharge from top, which is usually uniform, will lead to errors Thus recharge from top and bottom need separate treatment Special treatment is also needed for streams with clogged bed bounding the aquifer Usually beds and banks of streams are overlain by a thin semi-pervious layers called clogging layers (*fig*-3.1e) Presence of such a layer will affect the flow rate to or from stream considerably and hence the water table heights

3.2 GOVERNING PARTIAL DIFFERENTIAL EQUATION FOR FLOW IN UNCONFINED AQUIFERS

The governing partial differential equation for ground water flow in an unconfined aquifer consisting of homogeneous and isotropic nondeformable porous media can be easily derived by combining continuity equation with Darcy's law Consider the control volume shown in the (*fig*-3.2) From continuity,

$$\frac{\partial q_x}{\partial x} + \eta \frac{\partial h}{\partial t} - \frac{(h_o - h)}{c} - w = 0 \qquad (3.1)$$

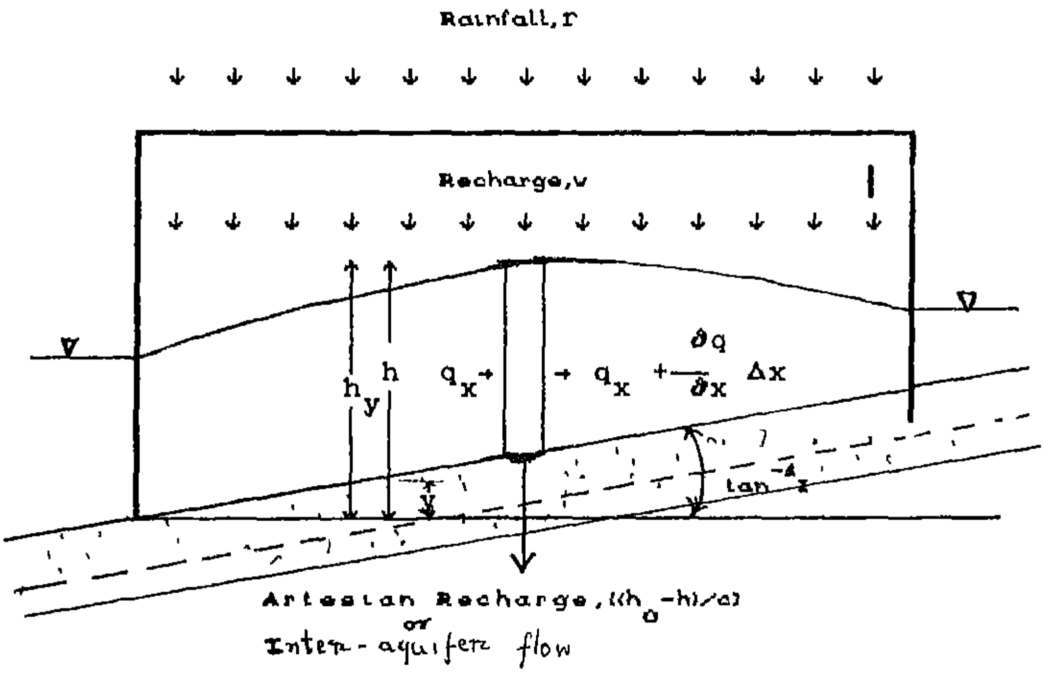
Where

q_x flow rate in positive x-direction,L^2T^{-1},

η effetive porosity,dimensionless,

x horizontal distance measured from origin,L;

t time,T;



Underlying aquifer

Fig.(3.2) Definition sketch for saturated ground water flow

c resistance of underlying semi-pervious layer, T,

h_o vertical distance to piezometric head in the underlying aquifer measured from bed, L,

h vertical height to phreatic surface measured from bed, L;

ω transiant recharge from top, LT^{-1}.

From Darcy's law

$$q_x = -k_s h \frac{\partial h_y}{\partial x} \qquad (3.2)$$

where

k_s saturated permeability, LT^{-1};

h_y verticle distance to the phreatic surface measured from datum, L;

I slope of bed, dimensionless

From (fig-3 2)

$$h_y = h + y \qquad (3\ 3)$$

Therefore,

$$q_x = -k_s h \left(\frac{\partial h_y}{\partial x} + I\right) \qquad (3\ 4)$$

Combining eqn (3 1) and eqn (3 4) we get the final equation as

$$\eta \frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left[k_s h \left(\frac{\partial h}{\partial x} + I\right)\right] + \frac{(h_o - h)}{c} + w \qquad (3\ 5)$$

3.2 1 ASSUMPTIONS INVOLVED

The above derivation involves following assumptions

(1) The acquifer is homogeneous and isotropic.

(2) The acquifer consists of nondeformable media.

(3) Darcy's law is valid.

(4) Dupit-Forchheimer assumptions are valid

(5) Recharge rate from top is small enough so that it gets refracted in the direction of groundwater flow itself.

(6) Flow of water in the underlying semipervious . . is vertical and as soon as it enters the unconfined acquifer it gets refracted in the flow direction prevalinig in the unconfined acquifer

(7) Bed slope is small

(8) The flow boundaries are vertical

3.3 INITIAL AND BOUNDARY CONDITIONS

The solution of any partial differential eqation requires additional information in the form of boundary conditions to evaluate the arbitrary constants appearing therein For unsteady flow problems initial condition is also required to determine the solution uniquely

3.3 1 INITIAL CONDITION

Initial condition in its most general form can be written as

$$h = f(x) \qquad \text{at } t=0 \qquad (3.6)$$

If the field data is available, function f(x) can be evaluated by fitting a polynomial for all the points or by representing it by a series of parabolas each fitting three consecutive points of f(x) Thus in general the initial condition can be taken as,

$$f(x) = \sum_{i=1,3,5}^{n} f_i(x) = \sum_{i=1,3,5}^{n} (a_i x^2 + b_i x + c_i) \qquad (3.7)$$

$$x_i < x < x_{i+2}$$

$$i = 1,3,5 \ldots$$

$$x_1 = 0 \qquad \ldots \quad (3.8)$$

$$x_{n+2} = 1 \qquad \ldots \quad (3.9)$$

In the solution initial condition is needed only to carry out an integration of the form

$$\int_{0}^{l_x} f(x)(A \sin \alpha x + B \cos \alpha x)\, dx$$

3 3 2 BOUNDARY CONDITIONS

When the water surface elevation above the bed is known at the boundary i,e in the neighbouring stream , the boundary condition is said to be of Dirichlet type and is given by

$$h = h_1 \qquad \text{at } x = 0 \qquad (3\ 10)$$

$$h = h_2 \qquad \text{at } x = l_x \qquad (3\ 11)$$

If fully impervious bed is present at the boundary the boundary condition is of no flux Neumann type and is given by

$$\frac{\partial h}{\partial x} = 0 \qquad \text{at } x = 0 \qquad (3.12)$$

$$\frac{\partial h}{\partial x} = 0 \qquad \text{at } x = l_x \qquad (3\ 13)$$

If banks of the streams forming the boundary are clogged due to the presence of a clayey material of low permeability and practically no storage,the boundary condition will be of Cauchy type and will be given as

$$k_s \frac{\partial h}{\partial x} + k_s I + \frac{k_1}{b_1}\left(h_1 - h\right) = 0 \qquad \text{at } x = 0 \qquad (3\ 14)$$

$$- k_s \frac{\partial h}{\partial x} - k_s I + \frac{k_2}{b_2}\left(h_2 - h\right) = 0 \qquad \text{at } x = l_x \qquad (3\ 15)$$

All the three types of boundary conditions mentioned above can be represented as

$$-k_s \; b_1 \frac{\partial h}{\partial x} - b_1 \; I + k_1 \; h = k_1 h_1 \quad \text{at } x = 0 \qquad (3\ 16)$$

$$k_s \; b_2 \frac{\partial h}{\partial x} + b_2 \; I + k_2 \; h = k_2 h_2 \quad \text{at } x = l_x \qquad (3\ 17)$$

where

b_1 thickness of clogging layer at left boundary, L;

b_2 thickness of clogging layer at right boundary, L;

k_1 permeability of clogging layer at left boundary, LT^{-1},

k_2 permeability of clogging layer at right boundary, LT^{-1};

h_1 transient water surface in left channel, L;

h_2 transient water surface in right channel, L;

In eqn (3 16) $b_1 = 0$ gives rise to Dirichlet type boundary condition whereas $k_1 = 0$ will give rise to no flux Neumann type bondary condition. In case none of them are zero it represents Cauchy type boundayr condition. Similarily, with $b_2=0$ and $k_2= 0$ in eqn.(3 17) give rise to Dirichlet and no flux Neumann type boundary condition respectively.

3.4 RESISTANCE AND CONDUCTANCE

The inter-aquifer flow between unconfined and semi-confined aquifer has to experience certain resistance 'c' Reciprocal of this is termed as conductance 'ρ' and is given by

$$\rho = 1/c \qquad (3.18)$$

Thicker the separating layer between two aquifers or lower its permeability larger will be the vertical resistance to inter-aquifer flow In absence of this layer the flow has to still encounter certain resistance due to the unconfined aquifer Thus the total resistance to the inter-aquifer flow comprise of two parts

$$c = c' + c_u \qquad .(3\ 19)$$

each part being given by

$$c' = b' / k' \qquad (3\ 20)$$

$$c_u = d_a / k_s \qquad (3\ 21)$$

where

b' thickness of the semi-pervious layer separating the two aquifers, L,

k' permeability of the separating layer, LT^{-1},

d_a average depth of flow, L;

k_s saturated permeability of unconfined aquifer, LT^{-1};

c total resistance to inter-aquifer flow, T;

c' resistance to inter-aquifer flow due to unconfined aquifer, T,

c_u resistance to inter-aquifer flow due to separating layer, T

The vertical resistance resistance due to unconfined aquifer is thought of to be concentrated at the bottom of the aquifer.

The unit of resistance to flow is same as that of time e.g days

Alike the inter-aquifer flow, the flow between the unconfined aquifer and the neighbouring stream experiences resistance if the bank of stream forming boundary is clogged This resistance is given by

$$C_1 = b_1 \ / \ k_1 \qquad (3\ 22)$$

$$C_2 = b_2 \ / \ k_2 \qquad (3\ 23)$$

3.5 PIEZOMETRIC HEAD IN UNDERLYING AQUIFER

The piezometric head at any point in the underlying aquifer is the height to which the water will rise if a piezometer is introduced at that point in the aquifer. The piezometric line consists of a number of points representing the piezometric head in the underlying aquifer. In fact the aquifer will be connected at its ends to reservoirs or water bodies located usually at far distances on either side Under the steady state condition the piezometric head line will be given by

$$h_o = \frac{h_r - h_l}{L_{rl}} x + h_l \qquad \ldots (3.24)$$

where

h_l = water surface elevation in the left reservior, L;

h_r = water surface elevation in the right reservior, L;

L_{rl} = distance between the two reservoirs, L.

In case the flow is unsteady due to the fluctuation of water levels in the bounding reservoirs $h_o(x,t)$ can still be found. But under practical circustances such data are seldom known. Presence of tubewells etc. create additional complexities. Since we are intrested in a small region, the piezometric surface for practical purposes is well given by

$$h_o = \frac{h_{or} - h_{ol}}{l_x} x + h_{ol} \qquad \ldots (3.25)$$

where

h_{or} piezometric head at right end, L;

h_{oL} piezometric head at left end,L

This can be written as

$$h_o = px+q \tag{3 26}$$

Describing h_o as function of x only is not a limitation of the method presented in this study for finding a soluition for the phreatic surface Transient piezometric surface can also be treated The only limitation is that it should be a known function of space and time. This means that it should not be affected appreciably due to the inter-aquifer flow In practice the presence of a seperating layer of low permeability makes such an assumption valid

3.6 TRNSIENT UNSATURATED FLOW ABOVE THE WATER TABLE

The recharge to water table from top will take place either due to rainfall or excess irrigation or artificial recharge of water.This recharge may be assumed uniform but will be unsteady usually The recharge is assumed steady in steady state problems or for the sake of simplification in case of unsteady state problems, but it should be noted that even a steady rainfall will cause an unsteady recharge to the water table due to the action of unsaturated zone The presence of plant roots will further complicate the problem. Negative recharge i,e evaporation from water table will take place during interstorm periods which will again depend upon a number of meterological, hydrological and topographical parameters.

3.6.1 GOVERNING DIFFERENTIAL EQUATION FOR UNSATURATED FLOW

The complex process can be fairly approximated by a partial differential equation,usually called Richard's equation with an additional sink term added to it to represent the transpiration

loss The equation can be derived easily by combining Darcy's law with continuity equation as follows

According to darcy's law

$$q_z = - k_u \frac{\partial h_u}{\partial z} \qquad (3\ 27)$$

The unsaturated head is given by

$$h_u = p_u - z \qquad (3.28)$$

where

q_z rate of unsaturated flow in positive z-direction LT^{-1}

h_u piezometric head in unsaturated soil, L,

k_u unsaturated permeability, LT^{-1},

z vertical co-ordinate measured from surface vertically downwards, L,

p_u soil matrix potential ,negativeve in unsaturated soils, L

From continuity,

$$\frac{\partial q_z}{\partial z} + T_a + \frac{\partial \theta}{\partial t} = 0 \qquad (3\ 29)$$

where

θ moisture content of soil mass, dimensionless,

T_a rate of transpiration per unit depth, T^{-1}

t time, T

Combining the two, we get

$$- \frac{\partial}{\partial z} \left[k_u \frac{\partial h_u}{\partial z} \right] + T_a + \frac{\partial \theta}{\partial t} = 0 \qquad (3\ 30)$$

or, $$\frac{\partial}{\partial z} \left[k_u \frac{\partial p_u}{\partial \theta} \frac{\partial \theta}{\partial z} \right] - \frac{\partial k_u}{\partial z} - T_a = \frac{\partial \theta}{\partial t} \qquad (3.31)$$

The governing partial differential equation becomes

$$\frac{\partial}{\partial z}\left[\delta \frac{\partial \theta}{\partial z} \right] - \frac{\partial k_u}{\partial z} - T_a = \frac{\partial \theta}{\partial t} \tag{3.32}$$

where δ is called diffusivity of the soil which is the rate of flow due to p_u and is given by

$$\delta = k_u \frac{\partial p_u}{\partial z} \tag{3.33}$$

Thus the first term on left hand side represents flow due to soil matrix potential where as the second term represents flow due to gravity and the third term the loss of water per unit depth due to transpiration by plants.The term on righ hand side represents rate of change in storage. The rate of recharge to water table will be given by

$$w = - \delta \frac{\partial \theta}{\partial z} + k_u \Big|_{z=1_z} \tag{3.34}$$

where 1_z is the depth of water table from surface. The recharge to water table can be evaluated if the above equation can be solved.

3.6.2 ASSUMPTIONS INVOLVED IN DERRIVING THE UNSATURATED FLOW EQUATION

The above derrivation makes the following assumptions :

(1) Darchy's law is valid for unsaturated flow.

(2) Isothermal conditions exists in unsaturated flow zone.

(3) k_u and p_u are single valued functions of moisture content θ.

(4) Incompressible flow takes place through a nondeformable media.

(5) The air effects are negligible.

(6) The roots are distributed uniformly in the root zone.

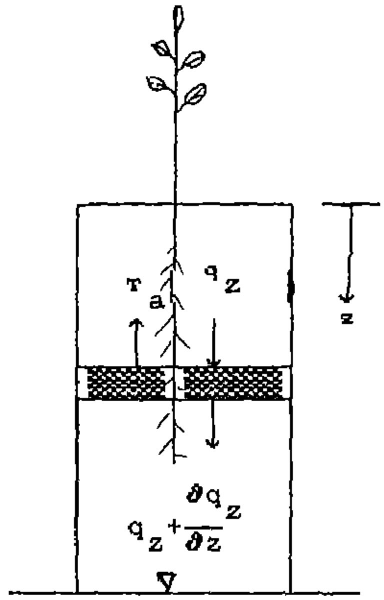


Fig 3 3 Definition sketch for unsaturated ground water flow

3 6 3 INITIAL AND BOUNDARY CONDITIONS FOR UNSATURATED FLOW

The boundary conditions at the top and at the water table as well as the initial condition are as follows

3 6 3 1 Initial Condition

The initial condition in its most general form may be written as

$$\theta = f(z) \qquad \text{at } t = 0 \qquad (3\ 35)$$

3 6 3.2 Boundary Condition At The Water Table

The boundary condition at the water table is always given by

$$\theta = \theta_s \qquad \text{at } z = l_z,\ t \geq 0 \qquad (3\ 36)$$

3.6.3 3 Boundary Condition At The Ground Surface

The boundary condition at the ground surface will vary with *time and to define it, the entire process is to be understood*

When the rainfall starts, initially the entire rainfall gets infilterated till the dry soil surface gets saturated. Then the rainfall in excess of infiltration gets accumulated on surface filling up the depressions and then run-off starts The run-off continues to take place till a short time after rainfall stops Then evapouration from the soil surface will consume the volume in detention storage alongwith infiltration During all this the soil surface remains saturated Thenafter desaturating of soil will occur due to the evapouration till the next rainfall takes place.

The boundary condition during the first phase prior to soil becomes saturated is given by

$$-\delta \frac{\partial \theta}{\partial z} + k_u = r \qquad t_{b1} \le t \le t_{s1} \qquad (3\ 37)$$

During the second phase soil surface remains saturated and the boundary condition is given by

$$\theta = \theta_s \qquad t_{s1} \le t \le t_{d1} \qquad (3\ 38)$$

In the third phase the rate of evaporation from soil surface can be found from metereological data and then the boundary condition is given by

$$\delta \frac{\partial \theta}{\partial z} - k_u = e \qquad t_{d1} \le t \le t_{b2} \qquad (3\ 39)$$

where

r transient rainfall rate, LT^{-1};

e potential evaporation rate, LT^{-1},

t_{b1} time instant corresponding to begining of first rainfall,

t_{b2} time instant corresponding to begining of next rainfall,

t_{e1} time instant corresponding to end of first rainfall;

t_{e2} time instant corresponding to end of next rainfall;

t_{s1} time instant corresponding to the time when surface gets saturated during first rainfall,

t_{d1} time instant corresponding to the time when the surface starts drying after first rainfall,

t_{r1} duration of first rainfall;

t_{c1} duration of one rainfall period which is the time since the begining of rainfall to the beginning of next rain;

In case the rainfall is not suffecient to saturate the soil surface, the phase I will change directly to III and phase II will be absent during that rainfall period The phase I boundary condition will prevail till t_{d1} and the phase III boundary condition will continue till t_{e2} Value of t_{s1} will be obtained during the solution but t_{d1} requires to be defined as the input data If volume in detention storage is assumed to be zero,

$$t_{d1} = t_{r1} \qquad (3\ 40)$$

Otherwise t_d can be found from following equation, if it is assumed that the volume in detention storage evaporates at the potential rate, which is given by

$$v_d = e\ (t_{d1} - t_{r1}) + \int_{t_{r1}}^{t_{d1}} i\ dt \qquad (3\ 41)$$

where i is the rate of infiltration given by

$$i = \left[-\delta \frac{\partial \theta}{\partial z} + k_u \right]_{z=0} \qquad (3\ 42)$$

This requires knowledge of detention volume (expressed as depth of water) and rate of evaporation under prevailing condition Later one is also required for defining boundary condition during third phase.

3 6 4 SOIL MOISTURE CHARACTERISTICS

The unsaturated soil moisture pressure p_u, unsaturated permeability k_u as well as diffusivity δ are all nonlinear functions of moisture content θ and hysterisis is exhibited. In the present study hysterisis will be neglected and it is proposed to use the average $\delta - \theta$, $p_u - \theta$, and $k_u - \theta$ curves.

Various investigators have investigated the $p_u - \theta$ and $k_u - \theta$ relationships and proposed a power form relationships as shown

in the table 3 1

TABLE 3.1

SOIL MOISTURE CHARACTERISTICS

REFERENCES	RELATIONSHIPS
Averjanov (1950)	$K = \phi^{3\ 5}$
Iramy (1959)	$K = \phi^{3.0}$
Brooks and corey (1966)	$K = \left[\frac{p_u}{p_b}\right]^{-(2+3\lambda)}$ $\phi = \left[\frac{p_u}{p_b}\right]^{-\lambda}$
Mualem (1976)	$K = [\phi]^{0\ 015\ W + 3\ 0}$ $W = \int_{\psi\to\infty}^{\psi\to 0} \gamma_w \psi \, d\theta$

Mualem's (1976) formula is based on generalised macroscopic approach and is found to be applicable for a wide variety of soils and moisture contents greater than wilting point It employes a macroscopic physical property which indicates the amount of work per unit volume of soil mass required to drain a saturated soil to the wilting point

Eagleson (1976 a) indicates that the power function model by Brooks and Corey (1966) applies nicely to the data of Talsma (1979) and Moore (1939) Also he indicates that a relationship between index of ϕ in K - ϕ relationship and λ , the pore size distribution index is valid for a wide variety of soils From the discussion it can be taken as

$$K = K_1(\phi)^{K_2} \tag{3 43}$$

$$P = P_1(\phi)^{P_2} \tag{3 44}$$

Therefore the diffusivity of the soil can also be represented by a power function relationship.

$$\Delta = D_1(\phi)^{D_2} \tag{3.45}$$

where

K non-dimensional unsaturated permeability, k_u / k_s,

P non-dimensional pressure head, p_u / p_b,

Δ non-dimensional diffusivity,

p_b bubbling pressure

3 6 5 ACTUAL TRANSPIRATION RATE

Corroborated results by Denmead and Shaw (1962) suggest that for a given potential rate there is a threshold average root zone soil moisture content (or the corresponding average soil moisture suction) below which the transpiration rate through the plant falls below the potential as indicated by Cordova and Brass(1981).

Neghasi (1974) suggested following relationship to relate the transpiration rate with soil moisture content,

$$t_a / t_p = \phi^{x} \qquad \phi < \phi^* \tag{3 46}$$

$$t_a / t_p = 1 \qquad \phi \geq \phi^* \qquad ..(3.47)$$

where

t_a Actual transpiration rate

t_p Potential transpiration rate

ϕ^* Threshold moisture content

Eagleson (1978b) describes a transpiration model as

$$t_a / t_p = (P-P^*)/\left[(r_s-r_p)\gamma_\omega T_{pv}\right] \qquad P < P^* \qquad (3\ 48)$$

$$t_a / t_p = K_v \qquad P \geq P^* \qquad (3\ 49)$$

where

r_s resistance to moisture flow in soils,

r_p resistance to moisture flow in plants,

γ_w specific weight of soil;

P^* critical leaf moisture potential,

K_v transpiration efficiency

The values of various coefficients can be found from plant physiological literature including the works of Evans (1963), Cowan (1968) Slatyer(1969) and Kramer (1969)

The transpiration model used in this study is given by

$$t_a / t_p = e'_1 \phi + e'_2 \qquad \phi_{wp} < \phi < \phi^* \qquad (3\ 50)$$

$$t_a / t_p = 1 \qquad \phi \geq \phi^* \qquad (3\ 51)$$

3.7 NUMERICAL METHOD USED FOR VARIFICATION OF RESULTS

Predictor - corrector method of Douglas and Jones (1963) has been used in this study to find numerical solutions, for both saturated as well as unsaturated flow .

The partial differential equation governing the saturated flow in nondimentional form is given by

$$\frac{\partial H}{\partial T} = H \frac{\partial^2 H}{\partial X^2} + \left(\frac{\partial H}{\partial X}\right)^2 + I \frac{\partial H}{\partial X} - C_o H + C_o H_o + W \qquad ..(3.52)$$

where H, X, T, S, C_o and U are non-dimensional quantities representing height of pheratic surface, horizontl distance, time, slope, vertical resistance and recharge respectively. The quantities with respect to which nondimensionalisation has been carried out depend on type of problem and hence will be discussed when a particular problem will be considered

Similarily, the equation governing unsaturated vertical flow in nondimensional form can be written as

$$\frac{\partial \phi}{\partial T} = \Delta \phi'' + \Delta'(\phi') - K' \phi' - e_1 \phi + e_2 \quad . \qquad (3\ 53)$$

Both the equations can be expressed as

$$\frac{\partial^2 U}{\partial X^2} = F\left(X, T, U, U', \frac{\partial U}{\partial T} \right) \qquad (3\ 54)$$

where U is the dependent variable, X is the independent variable and T is nondimensional time In both the cases 'F' can be written as

$$F = g_1(X,T,U,U')\frac{\partial U}{\partial T} + g_2(X,T,U,U') \qquad (3\ 55)$$

which is same as eqn.(1.9) of Douglas and Jones(1963).
Using Douglas Jones Predictor-Corrector method , the predictor is given by

$$0.5\ \Delta^2 X(U_{i,n+1/2} + U_{i,n}) = F\left[X_i,\ T_{n+1/2},\ U_{in},\ \delta_X U_{in},\ 0.5(U_{i,n+1/2} - U_{in})/\Delta T \right] \qquad (3.56)$$

The corrector is given by

$$0.5\Delta^2 X(U_{i,n+1}+U_{in}) = F\left(X_i, T_{n+1/2}, U_{i,n+1/2}, \delta_X U_{i,n+1/2}, \frac{U_{i,n+1}-U_{in}}{\Delta T}\right) \quad (3.57)$$

where
$$\Delta^2 X\, U_{in} = \frac{U_{i+1,n}-2U_{in}+U_{i-1}}{\Delta X^2} \quad (3.58)$$

$$\delta X\, U_{in} = \frac{U_{i+1,n}-U_{i-1,n}}{2\,\Delta X} \quad (3.59)$$

Thus a set of linear simultaneous equations are required to be solved for predicting the values of $U_{i,n+1/2}$ from which the values at the end of time step can be found by solving another set of linar simultaneous equations The value of H in case of steady state problems has been obtained as $T \to \infty$

CHAPTER 4

SOLUTION DISCUSSION AND APPLICATIONS

4.1 EXACT SOLUTION FOR STEADY STATE FLOW OVER HORIZONTAL SEMI-PERVIOUS BED

Leakage to or from the top most aquifer (we are interested in the flow in the top unconfined aquifer) may take place in addtion to recharge from top whenever the underlying bed is not impervious The quantity of artisen flow will be proportional to the difference of heads in underlying and unconfined aquifer Since the pheratic surface is a curved one,artisen leakage will not be uniform and hence cannot be treated as recharge from top which is usually taken as uniform Noticing this fact Wesseling and Wesseling (1986) have found a linearised solution An exact solution can be found using the solutions provided by Sikkema and Van Dam (1982), and Nieuwaal and Nijhuis (1974).

4.1.1 NON-DIMENSIONALISATION

The governing partial differential equation for steady state flow over horizontal semi-pervious bed is

$$\frac{\partial}{\partial x}\left(k_s h\frac{\partial h}{\partial x}\right) + \left(\frac{h_o - h}{c}\right) + \omega = 0 \qquad (4.1.1)$$

Using the non-dimensional parameters,

$$H=h/|h_e| \qquad . \qquad (4\ 1\ 2)$$

$$X=x/L_x \qquad . \qquad (4\ 1.3)$$

$$C_o = \frac{L_x^2}{K_e C |h_e|} \qquad (4\ 1\ 4)$$

$$\varepsilon = 1/C_o \qquad (4\ 1.5)$$

where h_e is the equivalent artesian head given by

$$h_e = (h_o + wc\ ) \qquad (4\ 1\ 6)$$

eqn (4 1 1) becomes

$$\frac{\partial}{\partial x}\left(H \frac{\partial H}{\partial x} \right) - C_o\ H \pm C_o \quad . \qquad (4\ 1\ 7)$$

(Use + when $h_e > 0$ and - when $h_e < 0$)

The solution of the above equation is found for following cases

(i) When heads at both boundaries are specified ,

(ii) When head and flux at one boundary are specified

4 1 2 SOLUTION

The eqn (4 1.7) can be written as

$$HH' \frac{d}{dH} (HH') = C_o\ H\ (\ H \pm 1\ ) \qquad (4\ 1\ 8)$$

$$(HH')^2 = \frac{2}{3}\ C_o (H^3 \pm 1\ 5\ H^2 + a) \quad . \qquad (4\ 1.9)$$

$$HH' = \sigma \sqrt{g(H)} \qquad .(4.1.10)$$

(Use -ve sign for $H_e > 0$ and +ve sign for $H_e < 0$)

where

$$\sigma = \sqrt{\frac{2}{3}\ C_o} \qquad (4.1\ 11)$$

and $g(H)$ is given by

$$g(H) = H^3 + \underline{1}.5H^2 + a \qquad (4.1.12)$$

The solution of eqn (4 1 9)depends on the value of 'a' which is governed by the relative elevations of artesian head line and water levels in channels forming boundary

4 1 2 1 When both the boundaries are specified

CASE-I WHEN $H_1 = H_2 = 1$

In this case $H = 1$ will be a solution of eqn (4 1 8) which means water table will be horizontal one with its height equal to $h_o+w\ c$ .Fig (4 1 1) shows the profile.

Before considering other three cases for $h_e>0$, $h_e=0$ and $h_e<0$ it is worthy considering flow in semi-infinite aquifers (fig-4 1 2 a,b,c,d)

When $h_e>0$ (fig-4.1.2 a,b)the height of phreatic surface changes from h_2 to h_e continuously and it meets the h_e-line tangentially at certain distance from boundary Boundary effects are confined in a distance l_{c2} only and beyond that water table is horizontal Selecting origin such that

$$X=0 \quad , \quad H=1 \quad , \quad \frac{\partial H}{\partial X} = 0 \qquad (4.1\ 13\ a)$$

$$X=L_{c2} \quad , \quad H=H_2 \qquad .(4.1.13\ b)$$

From eqn (4 1 9) with eqn.(4.1 13 a) we get

$$a=+0\ 5 \qquad (4\ 1\ 14)$$

The solution for this case follows from Nieuwaal and Nijhuis (1974) as

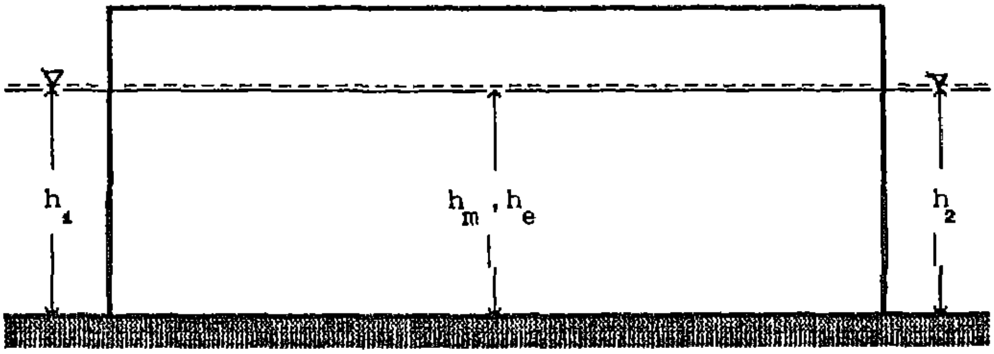


Fig 4 1 1 A typical steady state situation when water table is horizontal

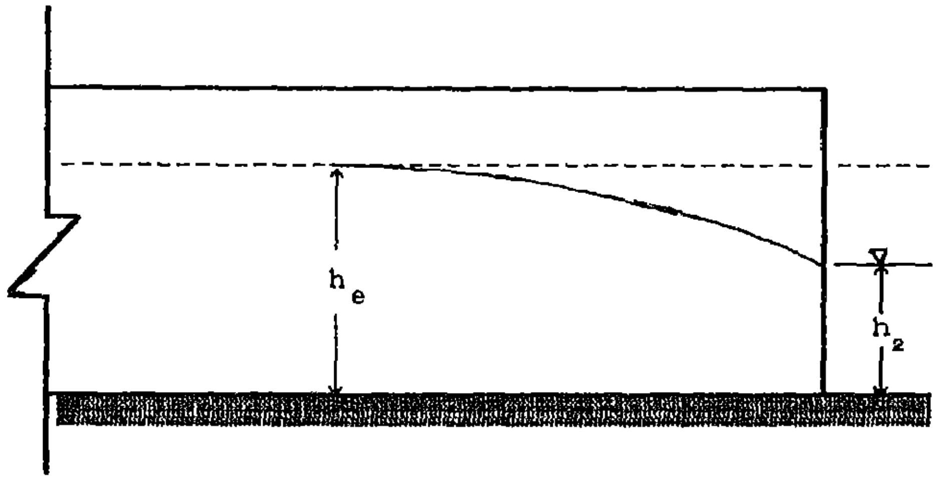


Fig 4 1 2 (a)

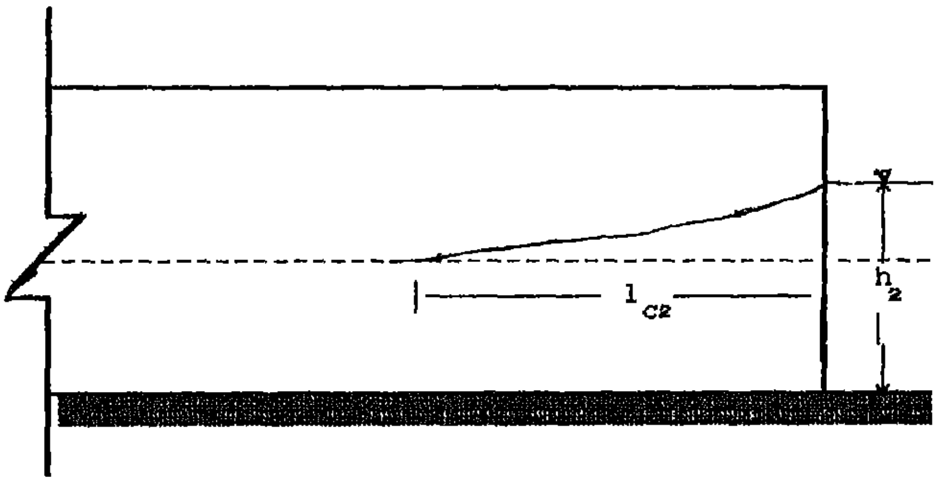


Fig. 4.1.2 (b)

Fig. 4.1.2 Flow over semi-pervious bed in semi-infinite domain

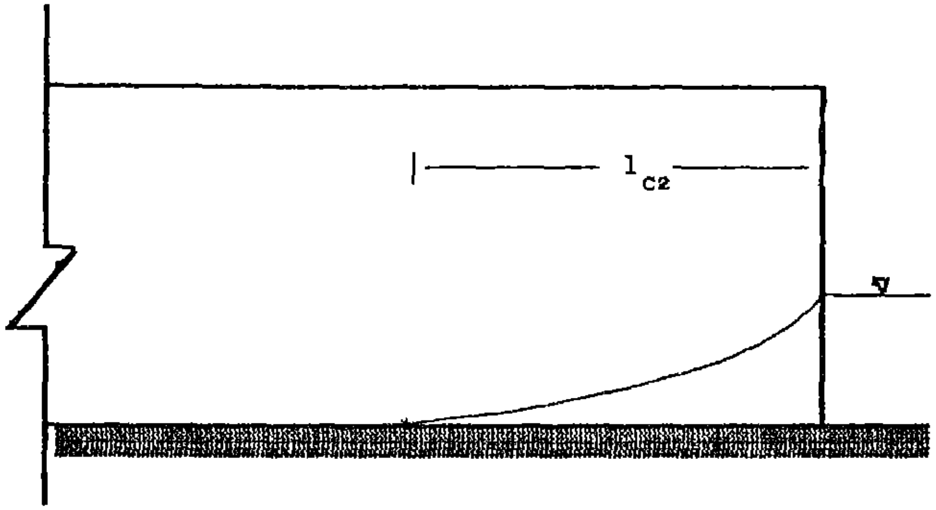


Fig 4 1 2 (c)

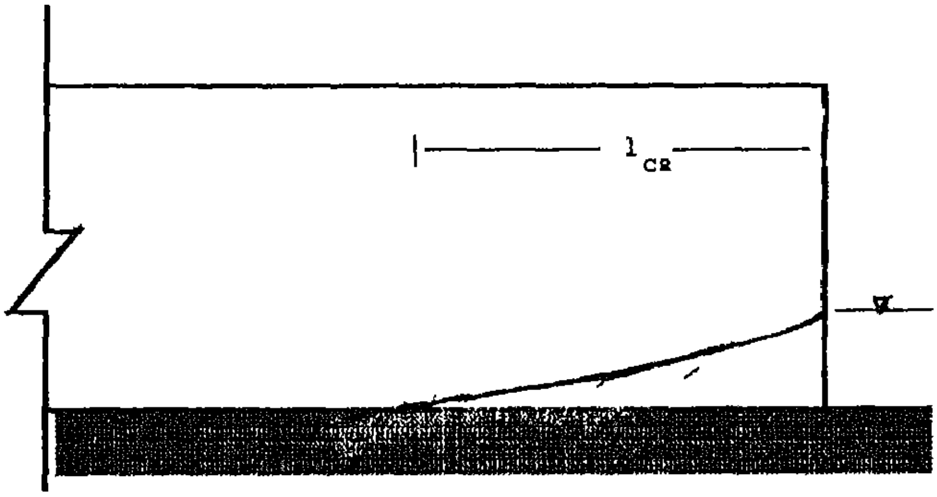


Fig. 4 1.2 (d)

Fig.4.1 2(d) Flow over semi-pervious bed in semi-infinite domain

$$\sqrt{H+0\ 5} \quad + \sqrt{1/6} \quad \ln \left| \frac{\sqrt{H+0\ 5} - \sqrt{1\ 5}}{\sqrt{H+0\ 5} + \sqrt{1\ 5}} \right|$$

$$= \sqrt{C_o / 6} \quad X + b \qquad (4\ 1\ 15)$$

where b is a constant of integration Since $h \rightarrow h_e$ asymptotically b is taken from x=0 at h=0 99h_e or 1.01 h_e Value of 'b' thus comes out to be 1 39 Now the profile can be computed from eqn (4 1 15) and l_{c2} is given by

$$l_{c2} = \sqrt{6k_a c\ h_e} \left[\sqrt{H_2 + 0.5} + \frac{1}{\sqrt{6}} \quad \ln \left| \frac{\sqrt{H_2 + 0\ 5} - \sqrt{1\ 5}}{\sqrt{H_2 + 0\ 5} + \sqrt{1.5}} \right| + 1\ 39 \right]$$

$$. \qquad (4\ 1\ 16)$$

When h_e<0 (*fig*-4 1 2 d) the height of phreatic surface changes continuously from h_2 to zero as it meets the horizontal bed at certain distance from the boundary Selecting origin such that

$$X=0 \quad , \quad H=0 \qquad (4.1\ 17\ a)$$

$$X=L_{c2}, \quad H=H_2 \qquad (4\ 1.17\ b)$$

From eqn.(4 1 9) and eqn.(4 1.17 a) we get

$$a=0 \qquad . (4\ 1\ 18)$$

The solution will be a parabola given by

$$H = \frac{C_o}{6} \left[X + \sqrt{9/C_o} \right]^2 - 1\ 5 \qquad ..(4.1.19)$$

where the second constant of integration 'b' has been evaluated

from$_c$ eqn (4 1 17 a) The profile can be found from eqn (4 1 19) and L_{c2} is given by

$$L_{c2} = \sqrt{6k_s ch_e} \left[\sqrt{1\ 5+H_2} - \sqrt{1\ 5} \right] \quad (4\ 1\ 20)$$

When he = 0 , non-dimensionalisation with respect to h_e will not be possible In fact in this case, using the dimentional variable itself and integrating equation

$$h\,h'' + h'^2 - h/c = 0 \quad (4\ 1\ 21)$$

we get,

$$\left(h \frac{\partial h}{\partial x} \right) = \sqrt{(2/3c)\ (h^3 + a)} \quad . \quad (4\ 1\ 22)$$

In this case (fig-4 1 2 c) the height of phreatic surface changes from h_2 at boundary to zero and it meets the horizontal bed tangentially Selecting the origin such that,

$$x=0 \quad , \quad h=0 \quad , \quad \partial h/\partial x=0 \quad . \ (4\ 1\ 23\ a)$$

$$x=l_c \quad , \quad h=h_2 \quad ..(4\ 1\ 23\ b)$$

From eqn (4 1 9) and eqn (4 1 23 a) we get,

$$a=0 \quad (4\ 1\ 24)$$

The profile in this case will be a parabola given by

$$h = \frac{x^2}{6c} \quad . \quad (4.1\ 25)$$

The second integration constant comes out to be zero Profile is

given by eqn (4 1 24) explicitly and L_{c2} is given by

$$l_{c2} = \sqrt{6\ h_2\ c} \qquad (4\ 1\ 26)$$

<u>CASE-II</u> WHEN $h_e > 0$

L_{c1} and L_{c2} are first found from eqn (4 1.16) for $h_e > 0$ If $1 \geq L_{c1} + L_{c2}$ the profile can be computed from eqn (4 1 15) Otherwise following cases arise

<u>Case-II A</u> When $1 > H_1 > H_2$

In this case (fig-4.1 3a) maximum water table height $h_m < h_e$ will occur somewhere in the flow domain. Selecting origin such that

$$X=0 \quad , \quad H=H_m \quad , \quad \frac{\partial H}{\partial X} = 0 \qquad (4.1\ 27\ a)$$

$$X=L_1 \quad , \quad H=H_1 \quad , \quad X=L_2 \quad , \quad H=H_2 \qquad (4\ 1\ 27\ b)$$

From eqn (4 1 9) and eqn (4 1 27 a) we get

$$a = -H_m + 1\ 5H_m^{\ 2} \qquad (4\ 1\ 28)$$

from which $0<a<0\ 5$

and therefore equation $g(H) = 0$ can be written as

$$g(H)=(H-r_1)(H-r_2)(H-r_3) \qquad (4.1.29)$$

Since $0<Hm<1$, $0<H<Hm$, the solution is given by

$$(r_3-r_1)\ E(\phi,\alpha) + r_3\ F(\phi,\alpha) = \pm\sqrt{\frac{(r_3-r_1)C_0}{6}}\ X + b \qquad (4\ 1\ 30)$$

where

$$\sin^2\phi = (H-r_1)/(H_m-r_1) \qquad (4\ 1.31)$$

$$\sin^2\alpha = (r_2-r_1)/(r_3-r_1) \qquad (4\ 1\ 32)$$

$$H_m = r_2 \qquad (4\ 1\ 33)$$

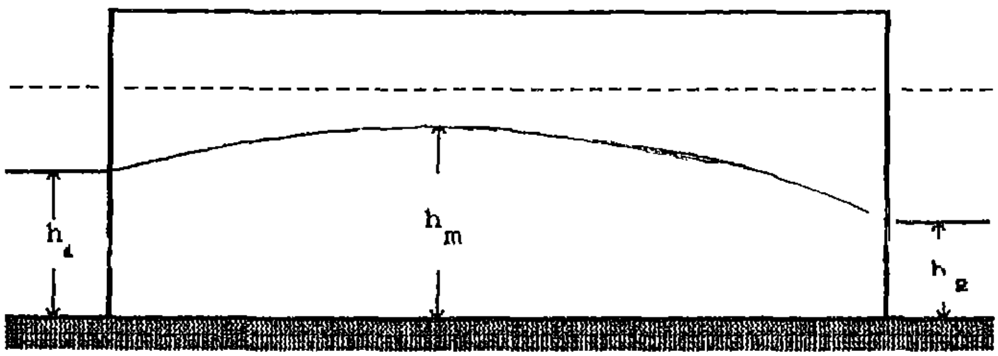


Fig 4 3.1 (a)

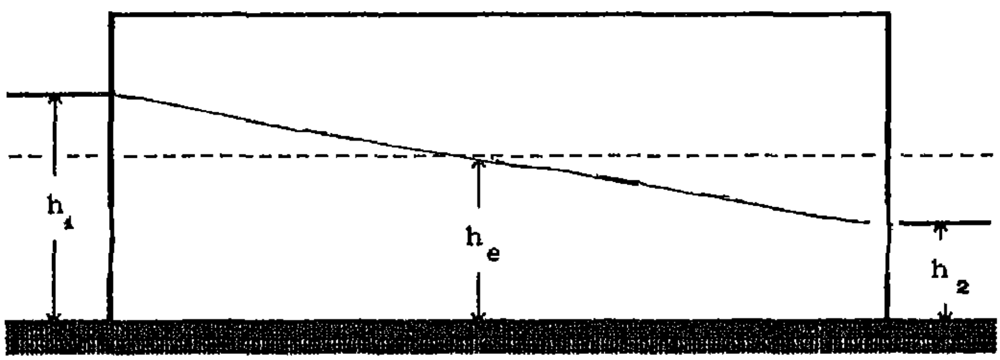
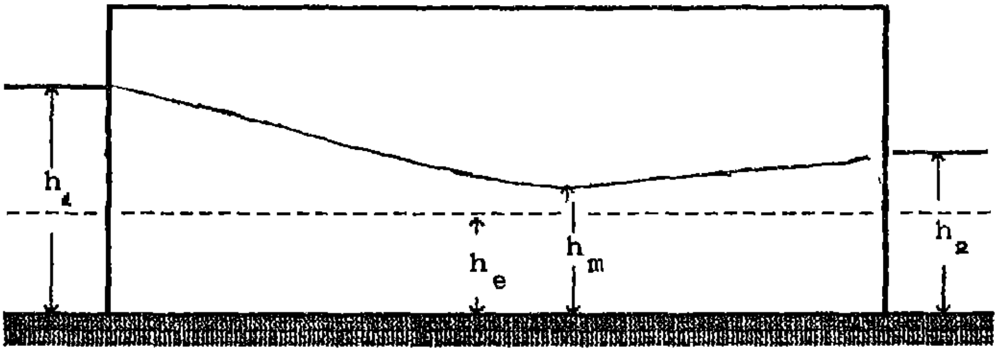


Fig. 4.3.1 (b)

Fig 4 1 3 Flow over horizontal semi-pervious bed due to a positive artesian head



(c)

Fig (4 1.3) Flow over a horizontal semi-pervious bed due to a positive artesian head

Fig 4 1 3 (contd.)Flow over horizontal semi-pervious bed due to a positive artesian head

and 'b' is an arbitrary constant of integration which can be evaluated by using equation (4 1 27 a) so that

$$b = (r_3 - r_1)E(90^o,\alpha) + r_3F(90^o,\alpha) \qquad (4.1\ 34)$$

E and F are elliptical integrals given by

$$E = \int_0^{\phi} \sqrt{(1-\sin^2\alpha)\ \sin^2\theta}\ d\theta \qquad (4\ 1\ 35)$$

$$F = \int_0^{\phi} \frac{1}{\sqrt{1-\sin^2\alpha \quad \sin^2\phi}}\ d\theta \qquad (4\ 1\ 36)$$

Case II B When $H_1 > 1 > H_2 > 0$

In this case (fig-4 1 3b) the pheratic line will meet h_e line somewhere but not tangentially Selecting origin such that

$$X=0 \quad , \quad H=1 \quad , \quad \frac{\partial H}{\partial X} = 0 \qquad .\ (4\ 1.37\ a)$$

$$X=L_1 \qquad H=H_1 \quad , \quad X=L_2 \quad , \quad H=H_2 \qquad (4\ 1.37\ b)$$

From eqn.(4 1 9) and eqn.(4 1 37 a) we get,

$$\frac{\partial h}{\partial X} = \pm \sqrt{\frac{2}{3}C_o} \quad \sqrt{a - 0\ 5} \qquad (4.1\ 38)$$

from which $a > 0\ 5$

and therefore equation $g(h)=0$ can be written as

$$g(h) = (h-r_1)\ (h^2 + 2ph + q) \qquad .\ .\ (4.1\ 39)$$

Since $\infty > h_1 > 1$ and $1 > h_2 > 0$ and noting $(q > p^2)$

the solution is given by following equation for either part

$$- 2\sqrt{A}\ E(\phi/\alpha) + \left[\sqrt{A} + \frac{r_1}{\sqrt{A}}\right] F(\phi/\alpha) +$$

$$+ 2\sqrt{A}\ \frac{\sin\phi\sqrt{1-K^2\sin^2\phi}}{(1+\cos\phi)} = \pm\sqrt{\frac{2C_\alpha}{3}}\ X + b \qquad (4\ 1\ 40)$$

where

$$\cos\phi = \frac{A + r_1 - h}{A - r_1 + h} \qquad (4.1\ 41)$$

$$A = \sqrt{r_1^2 + 2q} \qquad (4\ 1\ 42)$$

$$\sin^2\alpha = K^2 = \frac{A - r_1 - p}{2A} \qquad (4\ 1\ 43)$$

$$r_1 > 0$$

and b is a constant of integration which can be evaluated from eqn (4 1 37 a) and eqn (4 1 40) as

$$b = - 2\sqrt{A}\ E(\phi_1,\alpha) + (\sqrt{A} + \frac{1}{\sqrt{A}})\ F(\phi_1\alpha) +$$

$$+ 2\sqrt{A}\ \frac{\sin\ \phi_1\sqrt{1-K^2\sin^2\phi_1}}{(1+\cos\phi_1)} \qquad (4\ 1\ 44)$$

where ϕ_1 is given by

$$\cos\phi_1 = \frac{A + r_1 - 1}{A - r_1 + 1} \qquad . \quad (4.1.45)$$

Case II C When $H_1 > H_2 > 1$

In this case (fig-4 1 3 c) minimum water table height $h_m > h_e$ will occur somewhere in the flow domain Selecting origin such that

$$X=0 \quad , \quad H=H_m \quad , \quad \frac{\partial h}{\partial X} = 0 \qquad (4\ 1.46\ a)$$

$$X=L_2 \quad , \quad H=H_1 \quad , \quad X=L_2 \quad . \quad H=H_2 \qquad (4.1\ 46\ b)$$

From eqn.(4 1 9) and eqn (4 1 46 a),

$$a = H_m^2 \ (1.5 - H_m) \qquad (4\ 1\ 47)$$

When $H_m = 1\ 5$ In this case $a = 0$

and therefore the solution will be a parabola given by

$$H = \frac{C_o}{6} X^2 + 1.5 \qquad (4\ 1\ 48)$$

When $H_m < 1\ 5$ In this case $0\ 5 > a > 0$

so that $g(H)=0$ can be written as

$$g(H) = (H-r_1)(H-r_2)(H-r_3) \quad . \qquad (4.1\ 49)$$

and therefore the solution is given by

$$r_3\ F(\phi\ /\alpha) + (r_3 - r_1)\left[- E(\phi\ /\ \alpha) + \tan\phi\ \sqrt{1-K^2 \sin^2\phi}\ \right]$$

$$= \pm \sqrt{\frac{(r_3 - r_1)C_o}{6}}\ X + b \qquad (4.1\ 50)$$

where

$$\sin \alpha = 1 - \frac{r_a - r_2}{r_a - r_1} \qquad (4\ 1.51)$$

$$r_a = H_m \qquad (4\ 1\ 52)$$

$$\tan \phi = \sqrt{\frac{h - r_3}{r_3 - r_2}} \qquad (4\ 1\ 53)$$

and the second constant of integration 'b' can be evaluated from eqn (4.1 46 a) as

$$b = 0 \qquad (4\ 1\ 54)$$

When $H_m > 1\ 5$ In this case $a < 0$

so that equation g(H)=0 has one real root

and therefore the solution is given by

$$-2\sqrt{A}\ E(\phi / \alpha) + \left(\sqrt{A} + \frac{r}{\sqrt{A}}\right) F(\phi / \alpha) + \frac{2\sqrt{A}\sqrt{1-K^2\sin^2\phi}}{1 + \cos\phi}$$

$$= \sqrt{\frac{2\ C_o}{3}}\ X + b \qquad (4.1\ 55)$$

where

$$A = \sqrt{r_1^2 + 2q} \qquad (4.1.56)$$

$$\cos\phi = \frac{A + r_1 - h}{A - r_1 + h} \qquad (4\ 1.57)$$

$$K^2 = \sin^2 \phi \quad = \frac{A - r_1 - p}{2A} \qquad (4\ 1\ 58)$$

$$r_1 = H_m \qquad (4.1\ 59)$$

CASE-III WHEN $h_e < 0$

L_{c1} and L_{c2} are first found from eqn (4.1 20) for $h_e < 0$ If $1 \geq L_{c1} + L_{c2}$ the profile can be computed from eqn (4 1 19) Otherwise minimum water table height $h_m < h_e$ will occur somewhere in the flow domain Selecting origin such that

$$X=0 \quad , \quad H=H_m \quad , \quad \partial H/\partial X=0 \qquad (4.1\ 60\ a)$$

$$X=L_1 \quad , \quad H=H_1 \quad , \quad X=L_2 \quad , \quad H=H_2 \qquad (4\ 1\ 60\ b)$$

From eqn (4.1 9) and eqn (4 1 60 a),

$$a = H_m^2 (1\ 5 - H_m) \qquad . \quad (4\ 1\ 61)$$

When H_m=0 5 In this case (*fig*-4 1 4 a) $a = -0\ 5$

so that equation $g(H)=0$ can be written as

$$g(H)=(H+1)\ (H-0.5) \qquad (4.1\ 62)$$

and therefore the solution is given by

$$\sqrt{H-0\ 5} - \sqrt{2/3}\ \tan^{-1} \sqrt{2/3\ H-1/3} = \sqrt{C_0/6}\ X + b. \qquad (4.1\ 63)$$

where 'b' is a constant of integration which can be evaluated by using eqn (4 1 60 a) as

$$b = 0 \qquad . (4.1.64)$$

When H_m>0 5 In this case (*fig*-4.1.4 b) $a < -0.5$

so that equation $g(H)=0$ can be written as

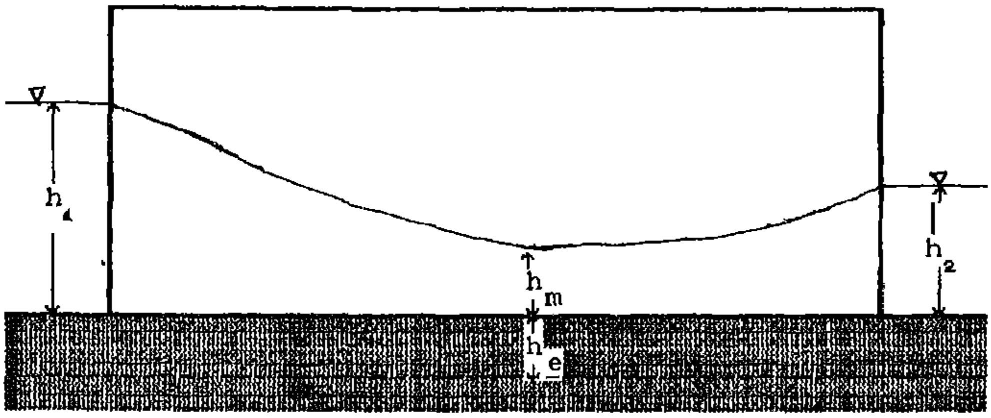


Fig.4 1.4 (a)

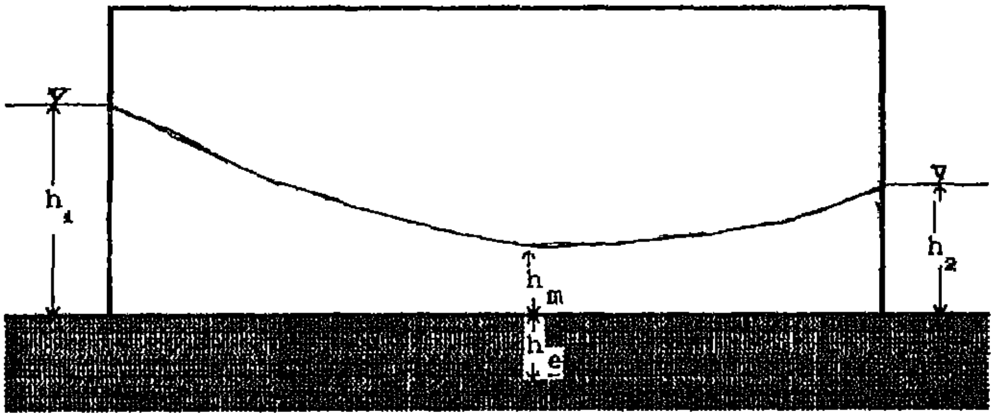


Fig. 4.1.4 (b)

Fig.4.1.4 Flow over horizontal semi-pervious bed due to a negative artesian head

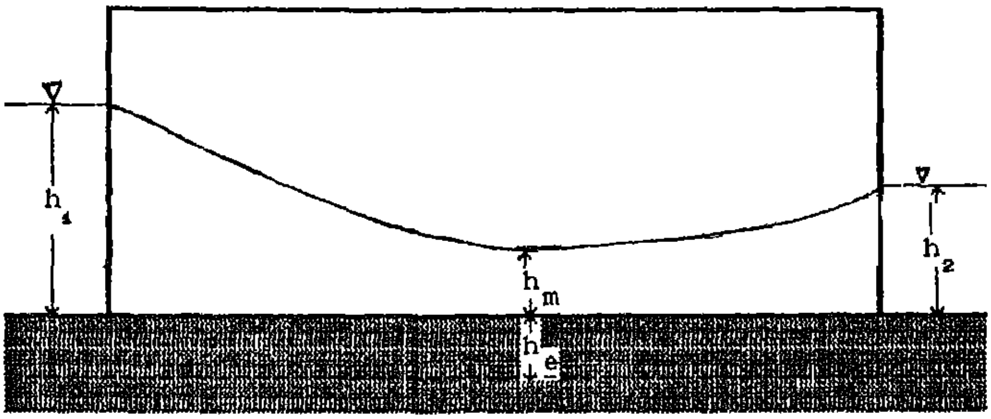


Fig. 4.1.4 (c)

Fig 4.1.4 (contd) Flow over horizontal semi-pervious bed due to a negative artesian head

$$g(H)=(H-r_1)\ (H^2+pH+q) \qquad (4\ 1\ 65)$$

and therefore the solution is given as

$$- 2\sqrt{A}\ E\ (\phi\ /\alpha\ ) + \left[\sqrt{A} + \frac{r_1}{\sqrt{A}}\right] F(\phi\ /\ \alpha) +$$

$$+ 2\sqrt{A}\ \frac{\operatorname{Sin}\phi\sqrt{1- K^2\operatorname{Sin}^2\phi}}{(1 + \operatorname{Cos}\phi)} = \pm\sqrt{\frac{2C_o}{3}}\ X + b \qquad (4\ 1\ 66)$$

where

$$\operatorname{Cos}\phi = \frac{A + r_1 - H}{A - r_1 + H} \qquad (4\ 1\ 67)$$

$$A = \sqrt{r_1^2 + 2q} \qquad .(4.1\ 68)$$

$$\operatorname{Sin}^2\alpha = K^2 = \frac{A - r_1 - p}{2A} \qquad .(4.1.69)$$

$$r_1 > 0$$

$$r_1 = h_m \qquad (4.1.70)$$

and 'b' is a constant of integration which can be evaluated from eqn.(4.1 66) and eqn (4 1 60 a) as

$$b = 0 \qquad (4\ 1\ 71)$$

WHEN $H_m < 0.5$ In this case (fig-4.1.4) $-0\ 5 < a < 0$ so that g(H) can be written as

$$g(H)=(H-r_1)(H-r_2)(H-r_3) \qquad . \qquad (4\ 1\ 72)$$

and $-1.5<r_1<-1$, $-1<r_2<0$ and $0<r_3<0.5$. Since H>1,the solution is given by

$$r_3 F(\phi,\alpha)+(r_3-r_1)[-E(\phi,\alpha)+\mathrm{tg}\,\phi\sqrt{1-K^2\sin^2\phi}$$

$$=\sqrt{\frac{(r_3-r_1)\,C_o}{6}}\quad X+b \qquad (4\ 1\ 73)$$

where

$$K^2 = 1-\sqrt{\frac{r_3-r_1}{r_3-r_2}} = \sin^2\alpha \qquad (4\ 1\ 74)$$

$$\tan\phi = \sqrt{\frac{H-r_3}{r_3-r_2}} \qquad (4\ 1\ 75)$$

$$r_3 = H_m \qquad (4\ 1\ 76)$$

The second constant of integration 'b' comes out to be zero

CASE-IV WHEN $h_e = 0$

l_{c1} and l_{c2} are first found from eqn (4 1 26) for $h_e > 0$ If $l_x \geq l_{c1} + l_{c2}$ the profile can be computed from eqn (4.1.25). Otherwise following cases arise ·

CASE-IV A WHEN $h_1>0$, $h_2>0$

In this case (fig-4 1 5 a) minimum water table height $h_m > 0$ will occur somewhere in the flow domain. Selecting origin such that

$$x=0 \quad , \quad h=h_m \quad , \quad \frac{\partial h}{\partial x} = 0 \qquad (4\ 1\ 77\ a)$$

$$x=l_1 \quad , \quad h=h_1 \quad , \quad x=l_2 \quad , \quad h=h_2 \qquad (4\ 1.77\ b)$$

From eqn (4 1.22) and eqn.(4.1.77 a),

$$a = -h_m^3 \qquad (4\ 1\ 78)$$

so that equation g(H)=0 can be written as

$$g(h) = (h-hm)(h^2+hmh+h_m^2) \qquad .. \quad (4\ 1\ 79)$$

and hence the solution is given by,

$$-2 \quad (3)^{\frac{1}{4}} E + \left((3)^{\frac{1}{4}} + (3)^{-\frac{1}{4}} \right) F(\phi,\alpha)$$

$$+ 2 \quad (3)^{\frac{1}{4}} \quad \frac{\sin\phi\sqrt{1-K^2\sin^2\phi}}{(1+\cos\phi)}$$

$$= \sqrt{\frac{2\ C_o}{3}} \quad X + b \qquad ....(4\ 1\ 80)$$

where

$$\alpha = 15^o \qquad ...(4\ 1.81)$$

$$\cos\phi = \frac{\sqrt{\sqrt{3}\ h_m + h_m - h}}{\sqrt{\sqrt{3}\ h_m - h_m - h}} \qquad ...(4.1.82)$$

and 'b' is a constant of integration which comes out to be

$$b = 0 \qquad ...(4\ 1.83)$$

CASE-IV B

In this case $\partial h/\partial x$ will not be zero anywhere in the flow domain since $l_x < l_{cz}$ .Selecting origin such that

$$x=0 \quad , \quad h=0 \qquad ..(4.1.84\ a)$$

$$x=l_x \quad , \quad h=h_2 \qquad ..(4.1.84\ b)$$

From eqn (4.1 22) and eqn.(4.1 84 a) $a=0$

and therefore the solution will be

$$h = \frac{x^2}{6c} \qquad ....(4.1.85)$$

From above it is seen that $\partial h/\partial x=0$ at $x=0$ which is not possible.This is because such a situation cannot occur as there will be seepage face at the other end

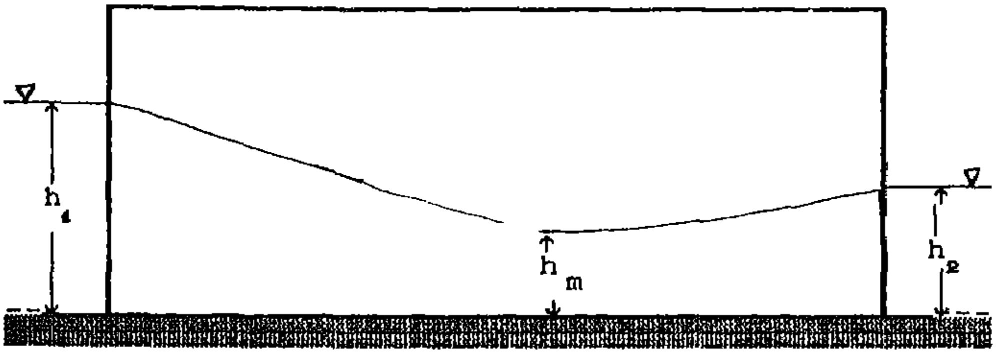


Fig 4 1 5 (a)

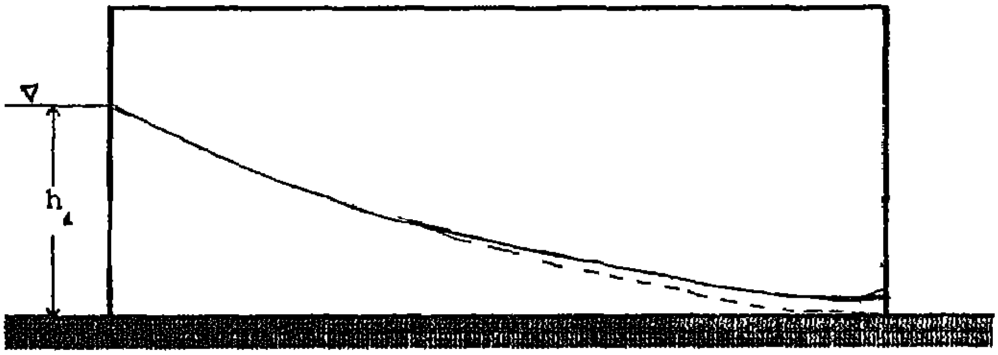


Fig 4.1.5 (b)

Fig.4 1 5 Flow over horizontal semi-pervious bed when artesian head is zero

Most of the solution found above involve independent variable x or X expressed as function of dependent variable h or $h/|h_e|$ When heads at both the boundaries are specified, a trial and error procedure is necessary to find h_m Once h_m is known, the profile can be easily computed

4 1 2 2 When Head and Flux are specified at one Boundary

When Head and Flux at one boundary are known, procedure is straight forward First constant a is found using h and $\frac{dh}{dx}$, from

$$a = \sqrt{\frac{3}{C_o} (hh')^2 - h^3 + 1\ 5h\ h_e^2} \quad . \quad (4\ 1\ 86)$$

Now depending on $h_e > 0$, $h_e < 0$, $h_e = 0$, and depending on the the values of h_1 ,h_e and a the proper case to which the problem belongs can be found from the last section From that solution, the profile can be plotted from the reference point to the boundary at which $h = h_1$, let this distance be l_1 Then the distance to the other boundary will be $(l_x - l_1)$ where l_x is the length of entire domain Putting $x = l - l_1$ in the solution will give us the value of h_2, thus profile at other side can also be plotted

4.3 1 APPLICATION TO DRAINAGE PROBLEM

4.1.3 1 Design and Analysis problem

Usually drains are employed to take away the artesian water whenever artesian head is high . This corresponds to case II A as $H_e > 0$.

In a design problem, spacing and depth of drains are to be found to maintain maximum water table at certain level . From

equation

$$l_x = \sqrt{6\ h_e\ k_s\ c}\quad I_n \tag{4 1.87}$$

where I_n is given by

$$I_n = -\left[(r_s - r_4)\ E(\phi\ /\ \alpha) + r_s\ F(\phi\ /\ \alpha) - b\right]\sqrt{6\ \ /(r_s - r_4)} \tag{4.1.88}$$

r, ϕ and α follows from eqn (4 1 29) to eqn (4.1 31) Assuming a suitable depth *of* drains 'd' , spacing of drains l_x can be found for a given h_m Thus l_x can be calculated for various values of d and a proper combination is selected The criterion for this selection may be economic In an analysis problem the maximum height of water table and water table profile is to be found, knowing 'd' the depth of drains and spacing 'l_x' . The procedure will be a trial and error one and is as follows :

1. Assume h_m and find h_m/h_e and d/h_e.
2 Find spacing and check whether it is correct or not If it is not correct,try another value of h_m Once h_m is found, rest of the profile can be easily computed

4 1 3 2 Comparison to Linearised Solution

Wesseling and Wesseling (1986) gave a linearised solution of eqn.(4 1.1) which in our notation after some rearrangement can be written as

$$H_m = 1 + (\ D - 1\ )\ /\ \cosh\ (0\ 5\ /\ \sqrt{\varepsilon\ D}\ ) \tag{4 1.88}$$

FIG (4 1.6) shows H_m calculated from linearised and exact solution for different depth of drains $d/\ h_e$ The dotted line shows curves for linearised solution and dashed lines for exact solution. The errors are plotted against d/h_m in (*fig*-4.1 7). Errors are large for small d/h_m It is seen that linearisation results in overestimation of h_m and this overestimation is larger for smaller values of ε It is found that error in h_m

found from linearised solution is less than 5% for D ≥ 0.5 and $\varepsilon \geq 0\ 2$ Also for D ≥ 0 2 maximum error in h_m is only 5.47% for $\varepsilon > 4$.

Thus it can be concluded that linearised solution works well. Since linearised solution is simpler one, it should be adopted for use Exact solution should be used when ε is less than 4 and only when D ≤ 0 5 .

4 1 3.3 Comparison with Young's Solution

Due to Dupuit's assumptions, the solution based upon it fails to account for convergence near a partially penetrating drain and seepage face Youngs(1986) found a solution which takes into account these factors The solution can be well described by following empirical formula

$$H_m = D + (1 - D)\left(1 - e^{-1\ 65\ L\ /\ 2\ h_o}\right). \qquad (4\ 1\ 89)$$

Youngs' formula does not take into account rainfall recharge from top and resistance of underlying stratum which in practice is found to occur usually Solution for this case using Dupuit's approach will be

$$I_m\ (H_m, D) = 1_x \,/\, \sqrt{2\ d\ h_o} \qquad . \qquad (4\ 1.90)$$

To account for convergence , Hooghoudt's equivalent depth 'D_{eq}' should be used instead of 'D' This is computed by using simple formula for Hooghoudt's equivalent depth given by Sakkas and Antonopoulos (1981) which are as follows .

$$D_{eq} = \frac{\sigma}{8\phi} \qquad (4\ 1\ 91)$$

$$\phi = \phi_1 + \phi_2 \qquad (4\ 1\ 92)$$

$$\phi_1 = \frac{-1}{\lambda \ln\left(\frac{rd}{h_o D}\right)} \qquad (4\ 1\ 93)$$

$$\phi_2 = 0\ 733 \log\sigma - 0.364 \qquad (4.1.94\ a)$$

$$0\ 01<\sigma<2$$

$$=0.345(\log\sigma)+0.508 \log\sigma - 0.326 \qquad (4\ 1\ 94\ b)$$

$$= 0\ 125\sigma + \frac{0.160}{\sigma} - 0\ 443 \qquad (4\ 1\ 94\ c)$$

where

$$\sigma = \frac{L}{D\ h_e} \qquad (4.1.95)$$

Using Young's formula, and exact and linearised formula without and in combination with Hooghoudt's equivalent depth, L/h_e and h_m/h_e for various values of D/h_e were calculated It was found that linearised solution in combination with Hooghoudt's equivalent depth compares with Young's solution nicely Assuming Youngs' solution as correct one, the errors in the linearised solution in combination with Hooghoudt's equivalent depth are in general less than 5 % though maximum error is 6.5 % in a particular case *FIG.* (4 1 8) shows the plot between h_m/h_o vs. L/h_o for various d/h_o for these two solutions Exact formula with Hooghoudt's correction gives better result when σ = 0 2 as maximum error is only 3 26% at L/h_o = 0 5. It is noted that at $d/h_o \leq 0.1$ the Dupuit - Forchmeir approach fails as flow will become more and more vertical Therefore it was concluded that linearised solution in combination with Hooghoudt's correction should be adopted for use This is because this formula can be used for a wider variety of problems when rainfall is there as well as when a semipermeable stratum separates the upper and lower aquifer which is commonly found to occur in practice.

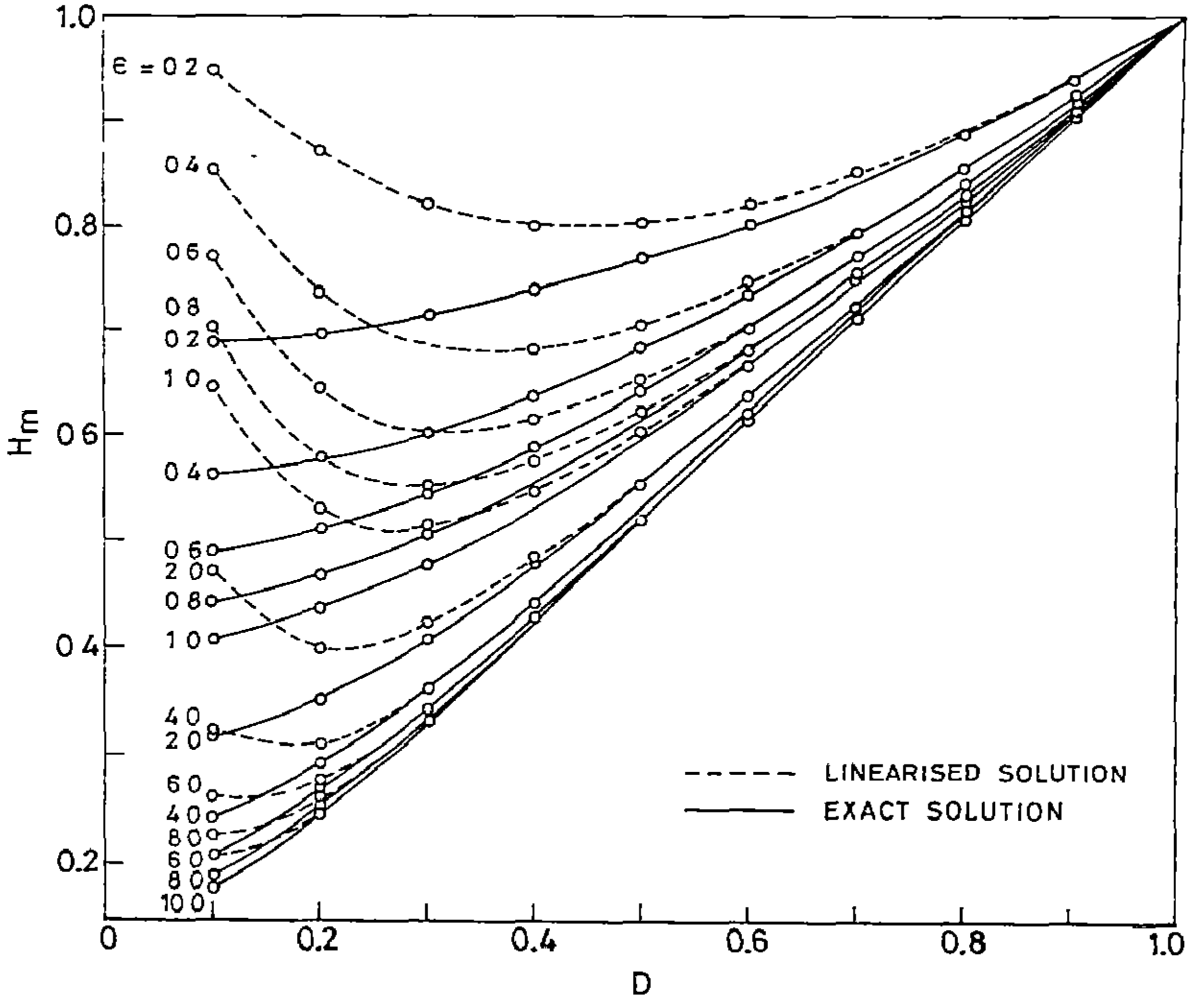


Fig. 4.1.6 Comparison of linearised and exact solutions.

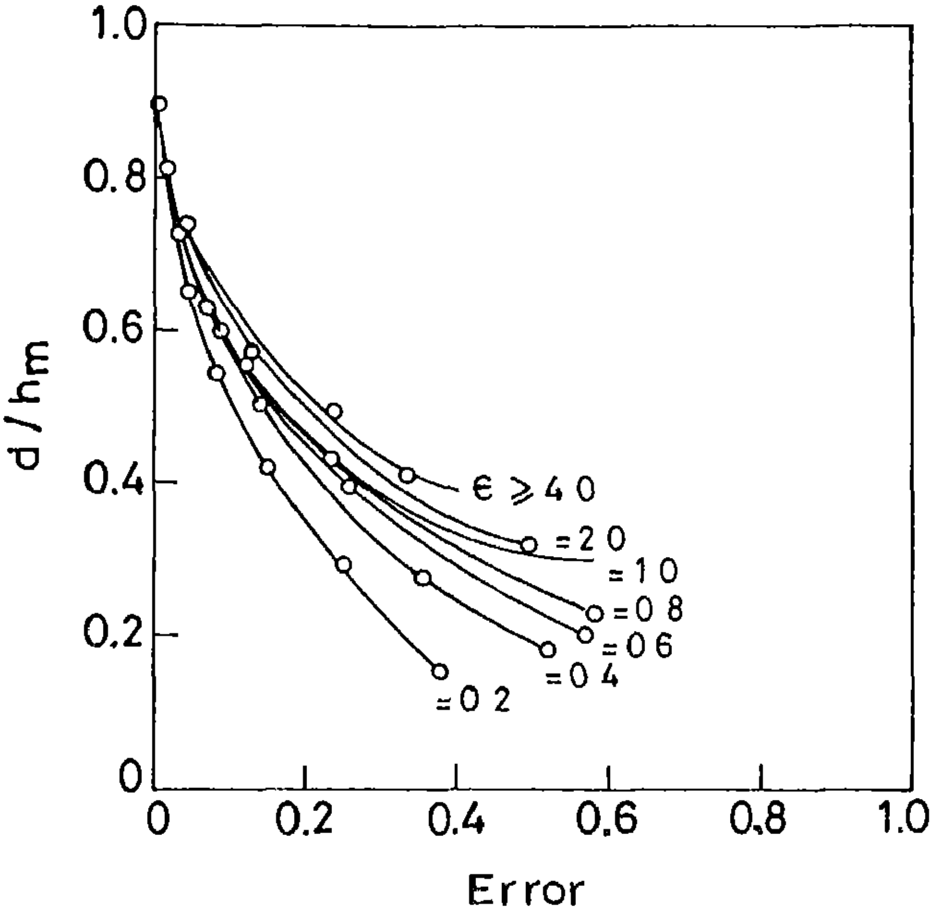


Fig.4.1.7 Errors due to linearisation.

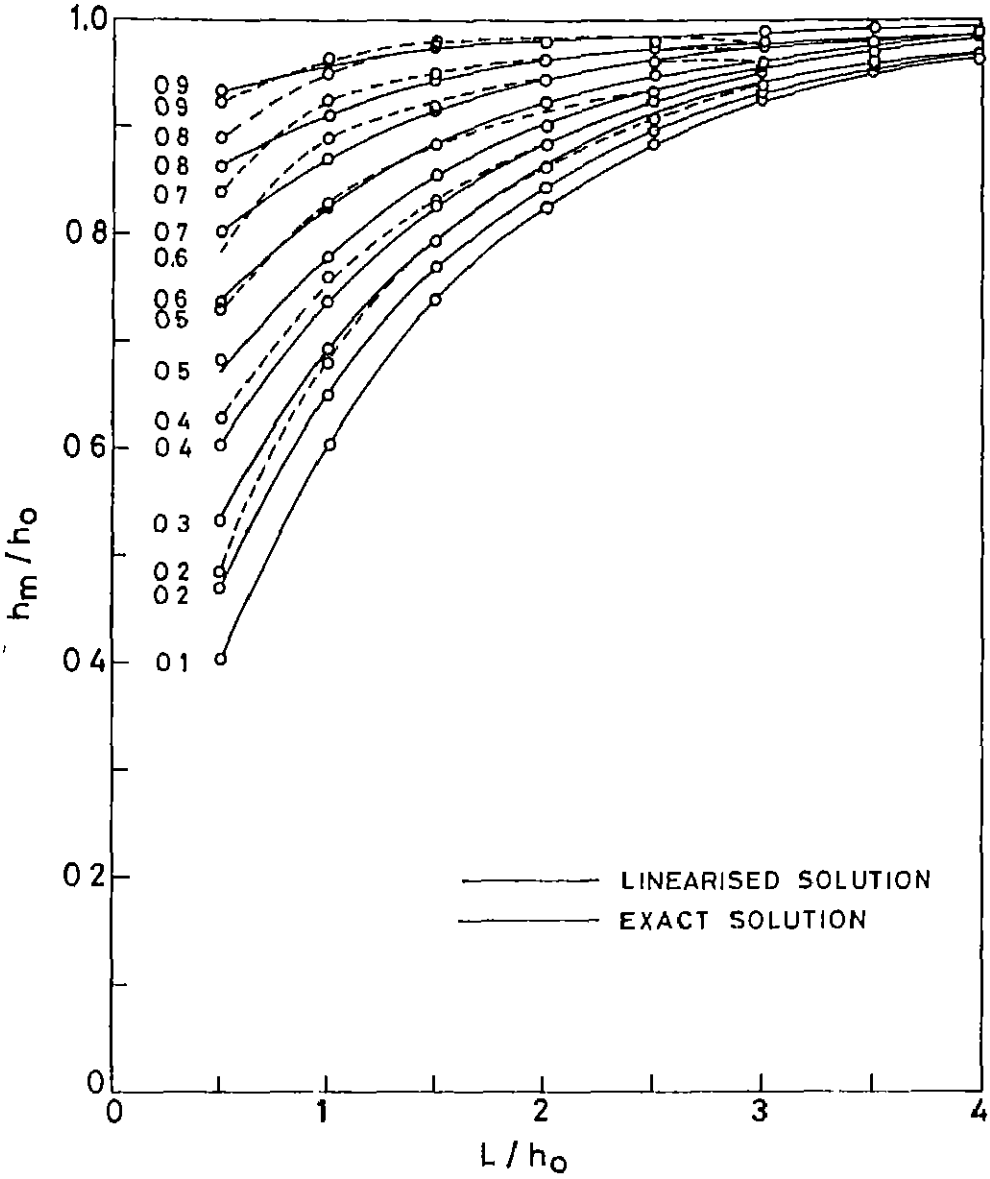


Fig 4.1 8 Comparison of linearised and exact solutions

TABLE - 4. 1.1

Errors due to linearisation for S = 0

	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
0.1	68.98	38.07	22.0	13.29	7.61	5.18	3.27	2.07	1.3	0.80
0.2	3.56	5.27	3.66	2.17	1.15	0.5	0.12	0.09	0.20	0.25
0.3	-1.86	-0.06	-0.7	-1.0	-1.0	-1.1	-1.02	-0.91	-0.8	-0.6
0.4	3.00	3.40	1.04	0.30	-0.7	-0.4	0.94	-0.8	-0.76	-0.6
0.5	8.5	6.13	3.41	1.30	0.21	0.26	-0.45	-0.5	-0.5	-0.4
0.6	7.5	7.35	4.41	2.30	0.98	0.30	0.40	-0.19	-0.2	-0.20
0.7	5.91	6.33	4.1	2.36	1.24	0.50	0.18	0.13	0.09	0.12
0.8	2.76	4.07	2.01	1.71	0.94	0.47	0.17	0.30	0.40	0.18
0.9	-1.16	0.91	0.95	0.59	0.30	0.11	0.0	-0.07	0.070	0.09

4.2 LINEARISED SOLUTION FOR STEADY STATE FLOW OVER SLOPING SEMI-PERVIOUS BED

The steady state flow over a sloping semi-pervious layer has been considered by Mualem and Bear(1978).They derived an anlytical solution by transforming the partial differential equation and then linaerising it. Their solution was only for a special case when artesian head was equal to $-b_o$,the thickness of sloping semipervious stratum In the present case arbitrary artesian heads are considered. The artesian head line will be assumed parallel to bed for flow between parallel drains and a linearised solution will be found The solution will be compared with the numerical solution to examine its validity.

4 2.1 NON-DIMENSIONALISATION

The partial differential equation applicable in this case will be

$$\frac{\partial}{\partial x}\left(k_s h \frac{\partial h}{\partial x}\right) + k_s I \frac{\partial h}{\partial x} + \frac{h_0 - h}{c} + \omega = 0$$

.(4.2.1)

$$h(0) = d \qquad (4\ 2.1\ a)$$

$$h(L) = d + I\ l_x \qquad .(4.2.1\ a)$$

Defining effectice artesian head as

$$h_e = h_o + c\ w \qquad (4.2.2)$$

and using non-dimensional parameters

$$H = h / h_e \qquad (4.2\ 3)$$

$$X = x / l_x \qquad (4\ 2\ 4)$$

$$\epsilon = K_s\ c\ h_e / l_x^2 \qquad (4\ 2.5)$$

$$S = I\ L / h_e \qquad (4\ 2.6)$$

we get

$$\frac{\partial}{\partial X}\left(H\, \partial H/\partial x\right) + S\,\frac{\partial H}{\partial x} - C_o H + C_o = 0 \qquad . (4\ 2.7)$$

with,

$$H(0)=D \qquad (4\ 2\ 7\ a)$$

$$H(1)=D+S \qquad (4.2\ 7\ b)$$

The exact solution of eqn (4.2.6) is difficult to find.

Hence linearisation approach becomes inevitable

4 2 LINEARISATION AND TRANSFORMATION

Eqn (4 2 6) can be linearised to get,

$$D\,H'' + H' - C_0 H + C_0 = 0 \qquad ..(4\ 2\ 8)$$

Using the transformation ,

$$H = U\ e^{-mX} \qquad .(4.2.9)$$

we get ,

$$DU''/n - U + C_0\, e^{mX}/n = 0 \qquad . (4\ 2\ 10)$$

$$U(0) = D \qquad (4\ 2\ 10\ a)$$

$$U(1) = (S+D)\ e^{-mx} \qquad (4\ 2\ 10\ b)$$

where
$$m = S/2D \qquad (4\ 2.11)$$

and
$$n = m\ D^2 \cdot C_0 \qquad (4\ 2\ 12)$$

4 2 3 SOLUTION

The solution of the above eqn (4 2.18) can be written as

$$U = H\ e^{mX} = C_1\ \mathrm{Sinh}\sqrt{\frac{n\ X}{D}} + C_2\ \mathrm{Cosh}\sqrt{\frac{n\ X}{D}} + e^{mX}$$

(4 2 13)

$$C_2 = D-1 \qquad (4\ 2.14)$$

and

$$C_1 = \frac{(D + S-1)\ e^{m} - (D-1)\ \mathrm{Cosh}\ \sqrt{n/D}}{\mathrm{Sinh}\ \sqrt{n/D}} \qquad . \qquad (4\ 2\ 15)$$

The maximum height of water table will occur where

$$\frac{\partial H}{\partial X} = 0 \qquad (4\ 2\ 16)$$

from which the distance to maximum height of water table X_m can be found to be given by the root of

$$\tanh X_m \sqrt{\frac{n}{D}} + \frac{m\ C_2 - C_1\sqrt{\frac{n}{D}}}{m\ C_1 - C_2\sqrt{\frac{n}{D}}} = 0\ . \qquad (4\ 2\ 17)$$

and then H_m will be given by

$$H_m = (\ C_1\ \mathrm{Sinh}\ X_m\ \sqrt{n/D} + C_2\ \mathrm{Cosh}\ X_m\ \sqrt{n/D}\ )\ e^{-mX} + 1 \qquad . \qquad (4\ 2\ 18)$$

4.2 3 COMPARISON WITH NUMERICAL SOLUTION

The maximum height of the water table h_m and its location X_m are computed from the eqn.(4.2 26) and eqn.(4 2 27)respectively and also by a numerical approach for a wide range of parameter ε varying from 0 2-10 The computation was carried out for five values of non-dimentional slope S The error in $h_m$the linearised solution was found and presented in tables From there it is seen that errors in H_m are small as long as $D \geq 0.4$ Errors in X_m are small but increase with ε and when ground water level reaches near upper drain errors become unacceptably large. Of course such situation will not be found in field as then the spacing will be uneconomicaly small. Thus for field situations linearised solutions are found to be acceptible

TABLE 4.2.1

ERRORS DUE TO LINEARISATION FOR S =0.05

D \ ε	0.2	0.4	0.6	0.8	1	2	4	6	8	10
0.1	43	62.4	69.5	68.2	63.2	36.2	5.39	-27.3	-16.9	-21.75
0.2	26.9	31	21.8	23.3	19	3.4	8.97	-13.7	-16.6	-18.1
0.3	15.72	14.68	11.45	8.39	5.85	-1.7	-6.45	-8.2	-8.93	-7.0
0.4	8.97	7.01	4.9	3.17	1.95	1.5	-3.4	-3.9	-4	-3.8
0.5	4.9	3.31	2.14	2	0.64	0.78	-1.44	-1.5	1.31	
0.6	2.47	1.5	0.81	0.44	0	0.44	-1.2	-0.76		
0.7	1.13	0.6	0	0	0	0				
0.8	0	0	0	0	0					
0.9	0	0	0							

TABLE 1 2 2

ERRORS DUE TO LINEARISATION FOR S =0 10

D \ \ ρ	0 2	0 4	0.6	0.8	1	2	4	6	8	10
0 1	41 66	60	67	65	63 11	40 61	10 01	-0 48	-7 8	-12 8
0.2	47 44	30 44	27.95	24 33	20 69	7 38	4 13	-8 14	-10 2	-11 2
0 3	13 62	14.86	12.07	9.34	7 35	0 64	-3 5	-4 33	-4 68	-4 6
0 4	8 95	7 24	5.41	3 79	2 59	-0 33	-1 57	-1 36	-0 91	
0.5	4 87	3 44	2.42	1 8	1 0	-19	0 36			
0.6	2 35	1 55	1.02	0.58	0	0 114				
0 7	1.35	0.6	1.12	0 18	0 16					
0.8	0 32	-0 16	5.83							
0.9										

TABLE 4 2 3

ERRORS DUE TO LINEARISATION FOR S =0 15

D \ ε	0 2	0 4	0 6	0.8	1	2	4	6	8	10
0.1	40 6	58 4	64 93	65.53	62 78	43 32	19.5	-1 15	0.16	-3 86
0 2	25 8	30	28 18	24.97	21.69	9 96	0 17	3 34	-4 7	-5.33
0 3	15.37	14.97	12.45	9.99	7 93	2.2	-1 32	-1 8		
0 4	8 86	7 39	5 47	4.13	2.95	0 39	-0 45			
0 5	4 65	3.54	2 41	1.6	1.11	0 01				
0 6	2 49	1 59	0.89	0 62	0.27					
0 7	1 06	0 58	1 28	2.35						
0.8										
0.9										

TABLE 4.2.4

ERRORS DUE TO LINEARISATION FOR S =0 20

D \ ρ	0 2	0 4	0.6	0 8	1	2	4	6	8	10
0 1	40	56 9	63 3	64.5	63.09	70 9	22 6	11 9	6 27	5 88
0 2	25 5	30	28 5	10.23	22	10.96	2 27	-0 68	-1 4	0 16
0 3	15.1	14.8	12 5	10.02	8.1	2 57	-0 3			
0 4	8.7	7.23	5 44	4.03	29	0 4				
0 5	4 6	3 27	2 25	1 45	0 93					
0 6	2 22	1 34	0.72	0 3	0					
0.7	0.8	0.22	0							
0 8										
0 9										

TABLE 4 2 5

ERRORS DUE TO LINEARISATION FOR S =0 25

D ε	0 2	0 4	0 6	0 8	1	2	4	6	8	10
0.1	36.5	51 3	56.9	58 6	56 5	50	30	17 7		
0 2	23.3	26 8	26 24	24 5	22 58	14 4	4 5			
0.3	14.1	13 77	12.2	10 4	8 8	3 6	0 16			
0.4	8.08	6.9	5.33	4 12	3 17	0 18				
0.5	4 28	2.96	1.94	-0 3	0 3					
0 6	1.93	0.85	0 12							
0 7	0 21									
0.8										
0 9										

4 3 EXACT SOLUTION FOR UNSTEADY STATE FLOW BETWEEN TWO PARALLEL DRAINS ON A HORIZONTAL SEMI-PERVIOUS BARRIER

The height of drains above the impervious bed have been used effectively for linearisation for all times but when the drains are on the barrier this is not pos'ible Realising this Glover (1954) found an approximate solution of governing partial differential equation without linerisation The solution involved an integral to be evaluated numerically In the present study an exact analytical solution will be found

4 3 1 NON-DIMENSIONALISATION

The mathematical problem is given by

$$\eta \frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(k_s h \frac{\partial h}{\partial x} \right) \qquad .(4.3\ 1)$$

with

$$h(x,0) = f(x) \qquad (4.3.1\ a)$$

$$\left.\frac{\partial h}{\partial x}\right|_{(0,t)} = 0 \qquad (4.3.1\ b)$$

$$h(l_x/2,t) = 0 \qquad (4.3.1\ c)$$

The co-ordinate system is shown in *fig* (4 3.1)

Using the non dimensional variables,

$$X = x/L \qquad (4.3.2)$$

$$H = h/h_m \qquad (4\ 3.3)$$

$$T = \frac{K_s\ t\ h_m}{\eta\ l_x^2} \qquad (4.3.4)$$

eqn. (4.3.1) becomes,

$$\frac{\partial H}{\partial T} = \frac{\partial}{\partial H}\left[H \frac{\partial H}{\partial X}\right] \qquad (4\ 3.5)$$

with

$$H(X,0) = F(X) \qquad (4\ 3\ 5\ a)$$

$$\left.\frac{\partial H}{\partial X}\right|_{(0,T)} = 0 \qquad (4\ 3\ 5\ b)$$

$$H(0\ 5,T) = 0 \qquad (4\ 3\ 5\ c)$$

where h_m is the maximum height of water table midway between the drains (i,e at X=0)at T=0

4 3 2 SEPERATION OF VARIABLES

Letting,

$$H(X,T) = F(X)\ G(T) \qquad (4\ 3\ 6)$$

We get,

$$\frac{1}{F}\frac{\partial}{\partial X}\left(F\frac{\partial F}{\partial X}\right) = \frac{1}{G^2}\frac{\partial G}{\partial T} = p \qquad (4\ 3\ 7)$$

where p is a constant Thus the following two differential equations are needed to be solved

$$F\ F'' + (F')^2 = p\ F \qquad (4\ 3\ 8)$$

with

$$\left.\frac{dF}{dX}\right|_{(0)} = 0 \qquad (4\ 3.8\ a)$$

$$F(0\ 5) = 0 \qquad (4\ 3\ 8\ b)$$

and

$$\frac{dG}{dT} = p\ G^2 \qquad (4\ 3\ 9)$$

with

$$G(0) = 1 \tag{4 3 9 a}$$

It should be noted that only h_m needs to be specified initially as F(X) is found from the solution This is because eqn.(4 3 6) indicates that profile at any time can be obtained from that of other time by multiplying it with a constant

4 3 3 SOLUTION

Eqn (4 3 8) can be written as

$$FF' \frac{d}{dx}(FF') = pF^2 \tag{4 3 10}$$

Integrating,

$$(FF')^2 = \frac{2}{3} pF^3 + a \tag{4.3 11}$$

and using eqn (4 3.8a)

$$a = - \frac{2}{3} p \tag{4 3 12}$$

Sinc F=1 at X=0 eqn (4.3 11) becomes

$$FF' = \pm \sqrt{\frac{2}{3} p(F^3 - 1)} \tag{4.3 13}$$

Since $0<h<h_m$, $0<F<1$ and hence p must be negetive to get the expression under the square root sign a positive quantity

Letting,

$$p = - q \tag{4.3.14}$$

eqn (4.3.13) becomes

$$FF' = \pm \sqrt{\frac{2}{3} q (1-F)(F^2+F+1)} \tag{4 3.15}$$

Integrating,

$$\int_1^F F (1-F)^{-0 5} (F^2+F+1)^{-0 5} dF = \pm \sqrt{\frac{2}{3} q} X + b \tag{4.3.16}$$

Rearranging,

$$- \int \sqrt{(1-F)/(F^2+F+1)}\ dF \quad + \int \sqrt{\frac{1}{(1-F)(F^2+F+1)}}\ dF$$

$$= \sqrt{\frac{2}{3}q}\ \ X + b \qquad (4\ 3\ 17)$$

Using,

$$\cos\phi = \frac{A - 1 + F}{A + 1 - F} \quad , A = \sqrt{3} \qquad (4\ 3\ 18)$$

and $K^2 = \sin^2\alpha = \dfrac{\sqrt{3} + 1 + 0.5}{2\sqrt{3}} = 0\ 9330127$ (4.3.19)

i e $\alpha = 75^{o}$ (4 3 20)

and noting that

$$\frac{d}{d\phi}\left[\frac{\sin\phi\sqrt{1-K^2\sin^2\phi}}{1+\cos\phi}\right] = \frac{1-\cos\phi}{1+\cos\phi}\quad\frac{1}{\sqrt{1-k\sin^2\phi}}$$

(4 3 21)

following solution of eqn (4 3 8) is obtained

$$2\,(3)^{\frac{1}{4}}\,E(\alpha,\phi)+\left(\frac{1}{(3)^{\frac{1}{4}}}-(3)^{\frac{1}{4}}\right)F(\alpha,\phi)-2\,(3)^{\frac{1}{4}}\,\frac{\sin\phi\sqrt{1-K^{2}\sin^{2}\phi}}{1+\cos\phi}$$

$$=\pm\sqrt{\frac{2}{3}q}+b \qquad (4\ 3\ 22)$$

when E and F are elliptical integral given by

$$E(\phi,\alpha)=\int_{0}^{\phi}\sqrt{1-k^{2}\sin^{2}\phi}\;d\phi \qquad (4\ 3.23)$$

$$F(\phi,\alpha)=\int_{0}^{\phi}(1-k^{2}\sin^{2}\phi)^{-0\ 5}\,d\phi \qquad (4\ 3.24)$$

and b is a constant of integration.Using X = 0,F = 1,ϕ = 0, we get,

$$b = 0 \qquad (4.3.25)$$

Using X = 0.5, F = 0,ϕ = 74.457732 and tables provided by Abramowitz and Stegun (1964) to evaluate E (75,74 457732) and F (75,74.457732),we get,

$$q = 4\ 4454231 \qquad (4.3.26)$$

Therefore the solution for F(X) becomes

$$2.632\ E(\alpha,\phi)-0.556\ F(\alpha,\phi)-2\ 632\sin\phi\ (1-k^{2}\sin^{2})^{0\ 5}(1+\cos\phi)^{-1}$$

$$=1.7215156\ \ X \qquad (4.3.27)$$

Solution of (4 3 9) is given by

$$G=\frac{1}{q\ T+C} \qquad (4.3.28)$$

With (4 3 9a) we get

$$C = 1 \qquad (4.3.29)$$

and hence G is given by,

$$G = \frac{1}{q\ T + 1} \qquad (4\ 3\ 30)$$

Hence the solution of eqn (4 3 1) is given by eqn (4 3 6) with eqn (4 3 27) and eqn (4 3.30).

4 3 4 NON - DIMENTIONAL PROFILE

The non-dimensional profile has been calculated by eqn (4 3 27) and shown in table (4.3.1) Profile at any time can be obtained by multiplying the h / h_m values by a constant given by eqn (4 3.30) Profile so obtained at various time steps has been shown in fig - 4 3 1

Comparison with Glover's solution,

$$H = F(X)\ G(Y) \qquad (4\ 3\ 31)$$

where F is given by

$$\sqrt{3}\ X = \int_0^w \frac{w.dw}{\sqrt{1 - w^3}} \qquad (4.3\ 32)$$

and

$$G = \frac{1}{4.5\ T + 1} \qquad (4.3\ 33)$$

shows that instead of 1 7215 and 4 4454, he has got $\sqrt{3}$ and 4.5 .Also this solution involves an integral to be evaluated evaluated numerlacally Note that in this solution origin is taken at left drain In his solution, Glover used initially horizontal water table, though the mathematical problem taken was same as in eqn.(4.3 1) From the curves provided by him, it can be seen that at T = 0.014 the profile is similar as given in table (4.3.1).

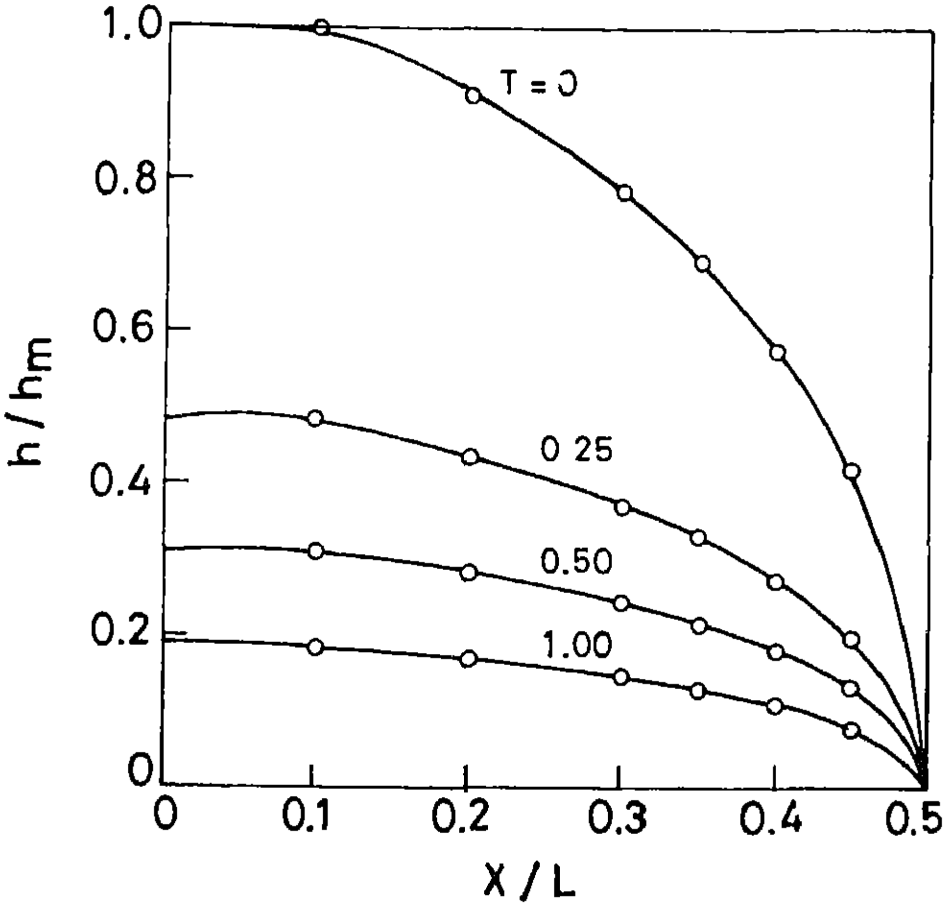


Fig. 4.3.1 Profile at various times for flow between two drains on the barrier.

TABLE 4.3.1

NON- DIMENSIONAL WATER TABLE PROFILE

X	H	X	H
0	1		
0.05	0.9997	0.30	0.822
0.10	0.9981	0.35	0.6924
0.15	0.9989	0.10	0.5734
0.20	0.9082	0.45	0.4142
0.25	0.8524	0.50	0

4.4 LINEARISED SOLUTION FOR TRANSIENT SATURATED FLOW OVER SLOPING SEMI-PERVIOUS BARRIER

The transient flow between two parallel channels constitutes one of the most important problems of ground water flow Exact solution is non-existent Linearised solutions for different cases were found by Werner(1957), Glover(1953), Moody and Tapp and Dumn (1960),Singh and Jacob (1974),Sewram and Chuhan (1967) for flows over fully impervious beds with Dirichlet type of boundary conditions Latinopoulas(1986) considered third type of boundary condition at one end, the boundary condition at other end being either no flux Neumann type (impermeable boundary) or Dirichlet type for flow over horizontal impervious bed with water table being horizontal initially Flow over horizontal semi-pervious bed was considered by Wessseling and Wesseling (1986) and they found a solution for flow between two drains when a steady rainfall occurs from top. In the present study a general solution will be found which is capable of handling all the three types of boundary conditions at either end for flow over a sloping semi-pervious bed

4 4.1 NON-DIMENSIONALISATION

The partial differential eqation governing unsteady flow over sloping semi-pervious bed is

$$\eta \frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left[k_s\, h\, \left(\frac{\partial h}{\partial x} + I\right)\right] + \frac{(h_o - h)}{c} + w \qquad (4.4\ 1)$$

With initial and boundary conditions as

$$h(x,0)=f(x) \qquad (4\ 4\ 1\ a)$$

$$-k_s\, b_1 \frac{\partial h}{\partial x} - b_1 k_s I + k_1\, h = k_1 h_1 \quad \text{at } x = 0\,. \qquad (4\ 4.1\ b)$$

$$k_s\, b_2 \frac{\partial h}{\partial x} + b_2 k_s I + k_2\, h = k_2 h_2 \quad \text{at } x = l_x \qquad (4.4\ 1\ c)$$

Using Nondimensional parameteres,

$$H = h/l_x \qquad (4.4.2)$$

$$X = x/l_x \qquad .(4.4\ 3)$$

$$W = w/k_s \qquad ..(4.4.4)$$

$$K_1 = k_1/k_s \qquad (4.4.5)$$

$$K_2 = k_2/k_s \qquad .(4.4\ 6)$$

$$T = k_s t/\eta l_x \qquad ..(4\ 4\ 7)$$

$$C_o = l_x /(k_s c) \qquad .\ (4.4.8)$$

$$B_1 = b_1/l_x \qquad (4\ 4.9)$$

$$B_2 = b_2/l_x \qquad .(4.4.10)$$

$$H_o = h_o/l_x \qquad .(4.4.11)$$

$$H_1 = h_1/l_x \qquad (4\ 4.12)$$

$$H_2 = h_2/l_x \qquad .(4.4.13)$$

eqn (4.4.1) becomes

$$\partial H/\partial T = \partial/\partial X(H\partial H/\partial X) + I\ \partial H/\partial X - C_o H + C_o H_o + W \qquad (4.4\ 14)$$

with initial and boundary conditions as

$$H(X,0) = F(X) \qquad (4\ 4\ 14\ a)$$

$$-B_1 \partial H/\partial X + K_1 H = K_1 H_1 + B_1 I \qquad (4\ 4\ 14\ b)$$

$$B_2 \partial H/\partial X + K_2 H = K_2 H_2 - B_2 I \qquad (4\ 4\ 14\ c)$$

4 4 2 LINEARISATION

As exact solution of eqn (4 4 14) is not possible the eqation is linearised following Glover (1953) as

$$\partial H/\partial T = D\ H'' + I\ H' - C_o H + C_o H_o + W \qquad (4\ 4\ 15)$$

with the initial and boundary conditionS remaining unchanged as

$$H(X,0) = F(X) \qquad (4\ 4\ 15\ a)$$

$$-B_1 \partial H/\partial X + K_1 H = K_1 H_1 + B_1 I \qquad (4\ 4\ 15\ b)$$

$$B_2 \partial H/\partial X + K_2 H = K_2 H_2 - B_2 I \qquad (4\ 4\ 15\ c)$$

4 4 3 TRANSFORMATION

Terms in linear eqation containing $\partial H/\partial X$ and H can be removed through following transformation

$$H = U\ e^{-(\mu X+\nu T)} + \int_o^T W\ dT \qquad (4.4\ 16)$$

so that we obtain

$$\partial U/\partial T = D\ U'' + (\ C_o H_o - C_o \int_o^T W\ dT\ )\ e^{(\mu\ X+\nu\ T)} \qquad \ldots.(4.4.17)$$

with the transformed initial and boundary condition as

$$U(X,0) = F(X)\ e^{\mu X+\nu T} \qquad (4\ 4\ 17\ a)$$

$$-B_1 \frac{\partial U}{\partial X} + (\mu B_1 + K_1) = [\ K_1 H_1 + B_1 I - K_{o} \int^{T} W\ dT\ ]\ e^{\nu T} \qquad (4\ 4\ 17\ b)$$

$$B_2 \frac{\partial U}{\partial X} + (-\mu B_2 + K_2) = [\ K_2 H_2 - B_2 I - K_{2\ o} \int^{T} W\ dT\ ]\ e^{\nu T} \qquad (4\ 4\ 17\ c)$$

where

$$\mu = I/2D \qquad (4\ 4\ 18)$$

$$\nu = \mu^2 D + C_o \qquad (4.4\ 19)$$

The nonlinear term in eqn.(4 4 17) can be removed through a further transformation

$$U = V + \psi\ e^{(\mu X+\nu T)} \qquad (4.4.20)$$

so that we get

$$\partial V/\partial T = D\ V'' \qquad (4\ 4\ 21)$$

where ψ is given by the following diffrential eqation

$$\partial\psi/\partial T = D\ \psi'' + I\ \psi' - C_o\psi + C_o H_o - C_{o\ o}\int^{T} W\ dT \qquad (4\ 4\ 22)$$

The solution for ψ depends on the form of $_o H\ (X,T)$ and $W(T)$ and can be found by assuming $\psi(X,T)$ to have a similar form as $H_o(X,T)$ in X

Thus for

$$H_o = P\ X + Q \qquad . \ (4\ 4\ 23)$$

asssuming $\psi(X,T)$ as

$$\psi = T_1(T)\ X + T_2(T) \qquad (4\ 4\ 24)$$

and putting in the above eqation for ψ we get two differential equationе as

$$\partial T_1/\partial T = -C_o T_1 + C_o P \qquad (4\ 4\ 25)$$

$$\partial T_2/\partial T = I\ T_1 - C_o T_2 + C_o Q - C_{o\ o}\int^{T} W\ dT. \qquad (4.4\ 26)$$

Solving eqn (4 4 25) we get T_1 as

$$T_1 = P \qquad (4\ 4\ 27)$$

Using this eqn (4 4 26) becomes

$$\partial T_2/\partial T + C_o T_2 = P\ I + C_o Q - C_o \int_o^T W\ dT \qquad (4.4.28)$$

$$T_2 = e^{-C_o T} \int (PI + C_o Q - C_o \int_o^T W\ dT) e^{C_o T}\ dT + C \quad e^{-C_o T} \qquad (4\ 4\ 29)$$

Selecting arbitrary constant C as zero and approximating recharge as

$$W(T) = \sum_{j=1,2,3}^{n} a_j\ T + b_j \qquad (4\ 4\ 30)$$

$$T_j < T < T_{j+1}$$

$$j=1,2,3.\ \ n$$

$$T_1 = 0 \qquad (4\ 4.31)$$

$$T_{j+1} = T_{end} \qquad ..(4\ 4.32)$$

the expression for T_2 becomes

$$T_2 = -\frac{a_j}{2} T^2 + \sigma\ T + \frac{PI + C_o Q - \sigma}{C_o} \qquad (4\ 4\ 33)$$

where

$$\sigma = \frac{a_j - C_o\ b_j}{C_o} \qquad (4\ 4\ 34)$$

Thus finally we obtain,

$$\partial V/\partial T = D\ \partial^2 V/\partial X^2 \qquad (4.4\ 35)$$

with initial and boundary conditions given by

$$V(X,0) = [F(X) - \psi(X,0)] e^{(\mu X)} \qquad (4.4.35\ a)$$

$$-B_1 \frac{\partial V}{\partial X} + (K_1 + \mu\ B_1) V = \left[K_1\ H_1 + B_1\ I - K_1\ {}_o\int^T W\ dT - K\ \psi(0,T) + B_1 \frac{\partial \psi}{\partial X} \right] e^{\nu T} \qquad (4.4.35\ b)$$

$$B_2 \frac{\partial V}{\partial X} + (K_2 - \mu B_2) V = \left[K_2 H_2 - B_2 I - K_2\ {}_o\int^T W\ dT - K_2 \psi(1,T) \right] e^{\mu + \nu T} \qquad (4\ 4\ 35\ c)$$

with the transforming function

$$V = (H-\int W dT-\psi)\ e^{\mu X+\nu T} \qquad (4\ 4\ 36)$$

The backtransformation is given by

$$H = V\ e^{-(\mu X+\nu T)} + \int_{0}^{T} W\ dT + \psi \qquad (4\ 4\ 37)$$

From eqn (4 4 24)

$$\frac{\partial\psi}{\partial X} = P \qquad (4\ 4\ 38)$$

Thus finally we want the solution of ,

$$\frac{\partial V}{\partial T} = D\ \frac{\partial^2 V}{\partial X^2} \qquad (4\ 4\ 39)$$

with initial and boundary conditions given by

$$V(X,0) = [\ F(X)-\psi(X,0)]e^{\mu X} \qquad (4\ 4\ 39\ a)$$

$$-B_1 \frac{\partial V}{\partial X} + E_1 V = G_1(T) \qquad (4\ 4\ 39\ b)$$

$$B_2 \frac{\partial V}{\partial X} + E_2\ V = G_2(T) \qquad (4.4.39\ c)$$

where

$$E_1 = K_1 + \mu\ B_1 \qquad (4.4.40)$$

$$E_2 = K_2 - \mu\ B_2 \qquad (4.4\ 41)$$

$$G(X) = [\ F(X)-\psi(X,0)]e^{\mu X} \qquad (4.4\ 42)$$

$$G_1 = (K_1 H_1 + B_1 I - K_1 \int_0^T W\ dT - K_1\ \psi(0,T) + B_1 P)e^{\nu T} \qquad (4\ 4\ 43)$$

$$G_2 = (K_2 H_2 - B_2 I - K_2 \int_0^T W\ dT - K_2 \psi(1,T) - B_2 P)e^{\nu T+\mu} \qquad (4\ 4.44)$$

The solution of above problem is given by ,

$$V = \sum_{N=1}^{\infty} \beta_N \gamma_{NX} e^{-D\alpha_N^2 \tau} (\varepsilon_N + D\alpha_N \tau_{NT}) + \zeta \qquad (4.4.45)$$

where α_N is the root of

$$\tan \alpha_N = \frac{\alpha_N (B_1 E_2 + B_2 E_1)}{B_1 B_2 \alpha_N^2 - E_1 E_2} \qquad (4.4.46)$$

$$\beta_N = \frac{2 B_2^2 \alpha_N^2 + E_2^2}{(B_1^2 \alpha_N^2 + E_1^2)[(B_1 \alpha_N^2 + E_1^2) + B_2 E_2] + B_1 E_1 (B_2 \alpha_N^2 + E_2)} \qquad (4.4.47)$$

$$\gamma_{NX} = B_1 \alpha_N \ \mathrm{Cos}\ \alpha_N X + E_1 \ \mathrm{Sin}\ \alpha_N X \qquad (4.4.48)$$

$$\varepsilon_N = \int_0^1 G(X)\, \gamma_{NX}\, dX \qquad (4.4.49)$$

$$\tau_{NT} = \int_0^T e^{D \alpha_N^2 \tau} [M_3 G_1 + M_4 G_2] \qquad (4.4.50)$$

$$\zeta = \int_0^1 G(X)\, dX \qquad \text{If } E_1 = E_2 = 0 \qquad (4.4.51)$$

$$= 0 \qquad \text{Otherwise} \qquad (4.4.52)$$

where

$$M_1 = B_1 \sin \alpha_N - \frac{E_1}{\alpha_N} (\cos \alpha_N - 1) \qquad (4.4.53)$$

$$M_2 = \frac{B_1}{\alpha_N} (\alpha_N \sin \alpha_N + \cos \alpha_N - 1) - \frac{E_1}{\alpha_N^2} (\alpha_N \cos \alpha_N + \sin \alpha_N) \qquad (4.4.54)$$

$$M_3 = L_1 M_1 - L_3 M_2 \qquad (4.4.55)$$

$$M_4 = L_2 M_1 + L_3 M_2 \qquad (4.4.56)$$

$$L_1 = E_1 (B_2 + E_2) / L_4 \qquad (4.4.57)$$

$$L_2 = B_1 E_2 / L_4 \qquad (4.4.58)$$

$$L_3 = E_1 E_2 / L_4 \qquad (4.4.59)$$

$$L_4 = B_4 F_1 + B_2 E_2 + E_3 E_1 \qquad (4\ 4\ 60)$$

To evaluate the initial condition integral ϵ_N the initial condition is approximated by a set of parabolas each fitting three consecutive points of F(X) as

$$F(X)=\sum_{i=1,3,5}^{m} a_i X^2 + b_i X + c_i \quad \text{for} \quad X_i \le X \le X_{i+2} \qquad (4\ 4\ 61)$$

$$X_1 = 0 \qquad (4\ 4\ 62)$$

$$X_{m+2} = 1 \qquad (4\ 4\ 63)$$

$$\epsilon_N = \sum_{i=1,2,3}^{m} \Big\{ [a_i X^2 + (b_i - T_1)X + c_i - T_{20}](B_1\alpha_N J + E_1 I) - [2a_i X + b_i - T_1] (B_1\alpha_N \int J + E_1 \int I) + [2a_i B_1 \alpha_N \int\int J + 2a_i E_1 \int\int I]\Big\}\Big|_{X_i}^{X_{i+2}} \qquad (4\ 4\ 64)$$

where

$$I = \int \sin \alpha_N X \quad e^{\mu X}\, dX \qquad (4\ 4\ 65)$$

$$J = \int \cos \alpha_N X \quad e^{\mu X}\, dX \qquad (4\ 4\ 66)$$

$$I = - \frac{e^{\mu X}}{\alpha_N^2 + \mu^2} (\alpha_N \cos \alpha_N X - \mu \sin \alpha_N X) \qquad (4\ 4\ 67)$$

$$J = \frac{e^{\mu X}}{\alpha_N^2 + \mu^2} (\mu \cos \alpha_N X + \alpha_N \sin \alpha_N X) \qquad (4\ 4.68)$$

$$\int I = - \frac{\alpha_N}{\alpha_N^2 + \mu^2}\, J + \frac{\mu}{\alpha_N^2 + \mu^2}\, I \qquad (4.4\ 69)$$

$$\int J = - \frac{\alpha_N}{\alpha_N^2 + \mu^2} I + \frac{\mu}{\alpha_N^2 + \mu^2} J \qquad (4.4.70)$$

$$\int\int I = - \frac{\alpha_N}{\alpha_N^2 + \mu^2} \int J + \frac{\mu}{\alpha_N^2 + \mu^2} \int I \qquad (4.4.71)$$

$$\int\int J = - \frac{\alpha_N}{\alpha_N^2 + \mu^2} \int I + \frac{\mu}{\alpha_N^2 + \mu^2} \int J \qquad (4.4.72)$$

To evaluate the boundary condition integral, the transient boundary condition is approximated as

$$H_1(T) = \sum (c_j T + d_j) \qquad (4.4.73a)$$

$$H_2(T) = \sum (e_j T + f_j) \qquad (4.4.73b)$$

$$T_j < T < T_{j+1}$$

$$j = 1, 2, 3 \ldots n$$

Then the boundary condition integral can be written as

$$\tau_{NT} = \sum_{j=1}^{n} (M_3 I_1 + M_4 I_2) \qquad (4.4.74)$$

where

$$I_1 = [A_1 T + B_1 - \varkappa A_1] \frac{e^{\varkappa T}}{\varkappa^2} \qquad (4.4.75)$$

$$I_2 = [A_2 T + B_2 - \varkappa A_2] \frac{e^{\varkappa T}}{\varkappa^2} \qquad (4.4.76)$$

$$A_1 = K_1(-b_j+c_j-\sigma) \tag{4 4 77}$$

$$A_2 = K_2(-b_j+c_j-\sigma) \tag{4 4 78}$$

$$B_1 = K_1(d_j - \frac{PI+C_o Q-\sigma}{C_o}) + B_1(I+P) \tag{4 4 79}$$

$$B_2 = K_2(f_j - \frac{PI+C_o Q^{o}_{-} \sigma}{C_o}) - B_2(I+P) \tag{4 4 80}$$

and $$\varkappa = D\,\alpha_N^2 - \nu \tag{4 4 81}$$

Thus boundary integral can be evaluated

4 4 5 APPLICATION TO A DRAINAGE PROBLEM

The solution obtained above has been applied to a problem involving flow over a sloping semi-pervious bed. The data and analytical results has been shown in fig-4 4 1

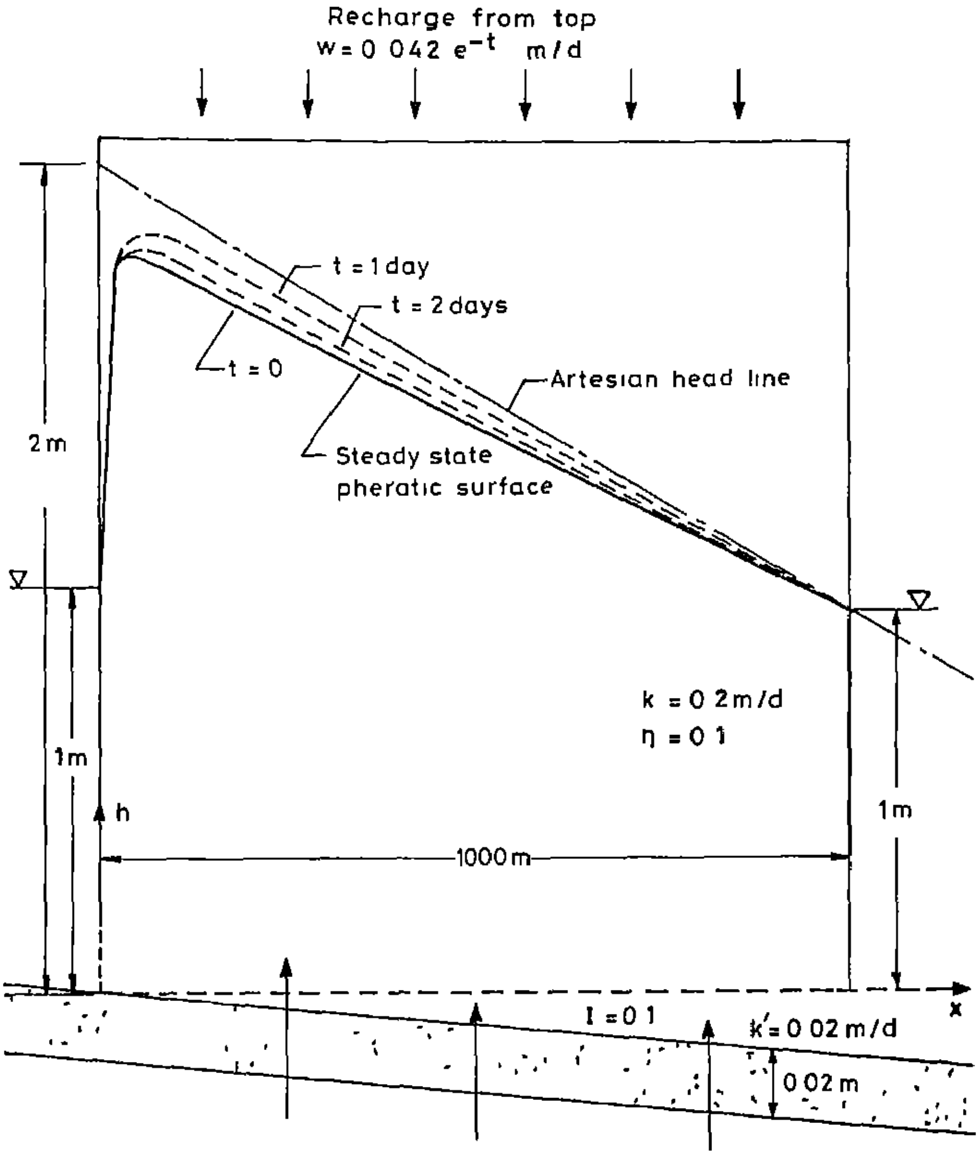


Fig. 4 4.1 Flow between parallel drains over sloping semi-pervious bed

4.5 LINEARISED SOLUTION FOR UNSATURATED FLOW ABOVE WATER TABLE

To asses the recharge to ground water table due to rainfall at ground surface requires the solution of Richards' equation for unsaturated flow Most of the solutions found till date are for a semiinfinite soil column with uniform initial saturation Some of them consider infiltration when the surface of soil is saturated whereas others assume a constant rate of infiltration at ground surface

4 5 1 GOVERNING DIFFERENTIAL EQUATION

The partial differential equation governing the transient unsaturated flow is

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left(\delta \frac{\partial \theta}{\partial z} \right) - \frac{\partial k}{\partial z} - \left(e_1 \theta + e_2 \right) \qquad (4\ 5.1)$$

with initial condition

$$\theta(z,0) = f(z) \qquad .\ (4\ 5\ 1\ a)$$

and boundary conditon at water table, $z = l_z$ as

$$\theta(l_z,t) = \theta_s \qquad .\ (4\ 5\ 1\ b)$$

and boundary condition at ground surface, $z = 0$ as

$$t < t_{s1} \qquad - \delta \frac{\partial \theta}{\partial z} + k_u = r \qquad .(4.5\ 1\ c)$$

$$t_{s1} < t < t_{d1} \qquad \theta = \theta_s \qquad (4.5\ 1\ d)$$

$$t_{d1} < t < t_{b2} \qquad - \delta \frac{\partial \theta}{\partial z} + k = -e \qquad (4\ 5\ 1\ e)$$

where

θ volumetric moisture content, non-dimensional,

θ_s saturation moisture content, non-dimensional,

z distance measured from ground surface vertically downwards,

l_z distance from ground surface to water table,

δ diffusivity of soil mass, L^2T^{-2},

k_u unsaturated permeability of soil mass, LT^{-1},

T_a actual transpiration rate per unit depth of root zone, T^{-1},

r rainfall rate, LT^{-1};

e potential evaporation rate, LT^{-1};

t time since start of rainfall, T,

t_{b1} time of begining of first rainfall, T,

t_{s1} time to saturation of soil surface since the beginning of rainfall, T,

t_{d1} time when drying of soil surface starts since the beginning of rainfall, T,

t_{b2} time to start of second rainfall since the beginning of first rainfall, T,

Using the non-dimensional parameters

$$\phi = \frac{\theta - \theta_r}{\theta_s - \theta_r} \qquad \text{.....(4 5 2)}$$

$$Z = z \,/\, l_z \qquad \text{. (4.5 3)}$$

$$T = \frac{k_s\ t}{l_z(\theta_s - \theta_r)} \qquad (4\ 5\ 4)$$

$$\Delta = \frac{\delta\ (\theta_s\ -\ \theta_r\ )}{k_s l_z} \qquad (4\ 5\ 5)$$

$$K = \frac{k_u}{k_s} \qquad (4\ 5\ 6)$$

$$R = \frac{r}{k_s} \qquad (4\ 5\ 7)$$

$$E_1 = \frac{l_z}{k_s}\ e_1(\ \theta s\ -\ \theta r\ ) \qquad (4\ 5.8)$$

$$E_2 = \frac{l_z}{k_s}\ (\ e_1\theta r\ -\ e_2\ ) \qquad (4\ 5\ 9)$$

where

θ_r residual moisture content,non-dimensional,

k_s saturated permeability of soil mass,LT^{-1}

the mathematical problem becomes

$$\frac{\partial\phi}{\partial T} = \frac{\partial}{\partial Z}(\ \Delta\ \frac{\partial\phi}{\partial Z}) - \frac{\partial K}{\partial Z} - (E_1\phi\ +\ E_2) \qquad (4.5\ 10)$$

with initial condition as

$$\phi(Z,0) = F(Z) = \frac{f(z)\ -\ \theta_r}{\theta_s\ -\ \theta_r} \qquad (4.5\ 10\ a)$$

and the boundary condition at water table, Z = 1 as

$$\phi(1,T) = 1 \qquad (4\ 5.10\ b)$$

and boundary condition at ground surface, Z = 0 as

$T_{b1} < T < T_{s1}$

$$-\Delta \frac{\partial\phi}{\partial Z} + K = R \qquad (4\ 5\ 10\ c)$$

$T_{s1} < T < T_{d1}$

$$\phi(0,T) = 1 \qquad (4\ 5\ 10\ d)$$

$T_{s1} < T < T_{d1}$

$$-\frac{\partial\phi}{\partial Z} + K = -E \qquad .(4\ 5\ 10\ e)$$

4 5 2 TRANSFORMATION

An exact solution of eqn (4 5 10) subjected to eqn (4 5 10a) eqn (4.5.10 b) and one of the eqn (4.5.10 c,d,e) is difficult because of its non-linearity and hence a linearisation approach becomes inevitable Such approaches have been used earlier,which consist of replacing Δ by Δ_{av} and $K \frac{\partial\phi}{\partial Z}$ by $K_{av} \frac{\partial\phi}{\partial Z}$ but were found to be of limited use

Eqn (4 5 10) can be written as

$$\frac{\partial\phi}{\partial T} = \Delta\,\phi'' + \Delta'\phi'^{2} - K'\phi' - E_1\phi - E_2 \qquad (4.5\ 11)$$

where

$$\phi' = \frac{\partial\phi}{\partial Z} \qquad (4.5\ 12)$$

$$\phi'' = \frac{\partial^2\phi}{\partial Z^2} \qquad (4.5\ 13)$$

$$\Delta' = \frac{\partial \Delta}{\partial \phi} \quad . \quad (4\ 5\ 14)$$

$$K' = \frac{\partial K}{\partial \phi} \quad (4\ 5\ 15)$$

Replacing Δ by Δ_{av} as used by earlier investigators makes term $\Delta' \phi'^2$ zero Instead we can use following transformation

$$\phi = F(U) \quad (4\ 5\ 16)$$

so that the transformed equation becomes

$$\frac{\partial U}{\partial T} = \Delta U'' - K' U' - \frac{\partial U}{\partial \phi} (E_1\phi + E_2) \quad (4\ 5\ 17)$$

with F(U) given by

$$\Delta F'' + \Delta' F'^2 = 0 \quad .(4.5.18)$$

Using following K-ϕ and Δ-ϕ relationships

$$K = K_1 \phi^{K_1} \quad . \quad (4\ 5\ 19)$$

and

$$\Delta = \Delta_1 \phi^{\Delta_2} \quad (4\ 5\ 20)$$

Eqn (4 5 18) can be solved and a particular selection of integration constants appearing in the solution yields

$$\phi = \frac{1}{U^{\Delta_2+1}} \quad .. (4\ 5\ 21)$$

$$U = \phi^{\Delta_2+1} \quad . .(4\ 5\ 22)$$

so that the eqn.(4 5 17) now becomes

$$\frac{\partial U}{\partial T} = \Delta U'' - K' U' - (\Delta_2+1)E_1 U - (\Delta_2+1)E_2 \quad . \quad (4.5.23)$$

From eqn (4 1 22)

$$\frac{\partial U}{\partial \phi} = (\Delta_2 + 1)\ \phi^{\Delta_2} \quad .(4.5\ 24)$$

Writing Δ, K and K' in terms of U

$$\Delta = \Delta_1 \; U^{\Delta_2/(\Delta_2+1)} \qquad (4\ 1\ 25)$$

$$K = K_1 \; U^{K_2/(\Delta_2+1)} \qquad . \quad (4\ 1\ 26)$$

$$K' = K_1 \; K_2 \; U^{(K_2-1)/(\Delta_2+1)} \qquad (4.5\ 27)$$

and therefore the eqn (4 5 17) becomes ,

$$\frac{\partial U}{\partial T} = \Delta_1 \; U^{\Delta_2/(\Delta_2 + 1)} \frac{\partial^2 U}{\partial Z^2} - K_1 \; K_2 \; U^{(K_2-1)/(\Delta_2+1)} \frac{\partial U}{\partial Z}$$

$$- (\Delta_2 + 1) \; E_1 U - (\Delta_2 + 1) E_2 \; U^{\Delta_2/(\Delta_2+1)}$$

(4 5 28)

with initial condition as

$$U(Z,0) = [\; F(Z)]^{\Delta_2+1} \qquad (4\ 5\ 29\ a)$$

and the boundary condition at water table, Z = 1 as

$$U(1,T) = 1 \qquad . \quad (4\ 5\ 29\ b)$$

and boundary condition at ground surface, Z = 0 as

$$T_{b1} < T < T_{s1} \qquad - \frac{\Delta_1}{(\Delta_2+1)} \frac{\partial U}{\partial Z} + K_1 \; U^{K_2/(\Delta_2+1)} = R$$

(4.5 29 c)

$$T_{s1} < T < T_{d1} \qquad U=1 \qquad (4\ 5.29\ d)$$

$$T_{s1} < T < T_{b2} \qquad - \frac{\Delta_1}{(\Delta_2+1)} \frac{\partial U}{\partial Z} + K \; U^{K_2/(\Delta_2+1)} = - E$$

. (4.5.29 e)

The advantages of tansformation given by eqn (4 5 16) are now evident The term containing Δ' has been taken into account Also now exact and linearised solutions of eqn (4 5.29) will differ less because powers of 'U' requiring linearisation are generally less than 1 When defining back the profile in terms of ϕ , U will have to be raised by power of $\frac{1}{\Delta_2+1} < 1$ always and hence the errors will further go down approximately by a by a factor of $\frac{1}{\Delta_2+1}$

4 5 3 LINEARISATION

The transformed unsaturated flow equation can be linearised so that the linearised equation becomes

$$\frac{\partial U}{\partial T} = L_1 U'' - L_2 U' - L_3 U - L_4 \qquad (4\ 5\ 29)$$

with initial condition as

$$U(Z,0) = (F(Z))^{\Delta_2+1} \qquad (4\ 5\ 29\ a)$$

and the boundary condition at water table, Z = 1 as

$$U(1,T) = 1 \qquad (4\ 5\ 29\ b)$$

and boundary condition at ground surface, Z = 0 as

$$T_{b1} < T < T_{s1} \qquad -L_5 \frac{\partial U}{\partial Z} + L_6 U = R \qquad (4\ 5\ 29\ c)$$

$$T_{s1} < T < T_{d1} \qquad U = 1 \qquad (4.5\ 29\ d)$$

$$T_{d1} < T < T_{b2} \qquad -L_6 \frac{\partial U}{\partial Z} + L_6 U = -E \qquad (4\ 5.29\ e)$$

where

$$L_1 = \Delta_1 U^{\frac{\Delta_2}{\Delta_2+1}} \qquad .(4.5.30)$$

$$L_2 = K_1 K_2 \, U^{\frac{K_2 - 1}{\Delta_2 + 1}} \qquad (4\ 5\ 31)$$

$$L_3 = (\Delta_2 + 1)\ E_1 \qquad (4\ 5\ 32)$$

$$L_4 = (\Delta_2 + 1) E_2 \, U^{\frac{\Delta_2}{\Delta_2 + 1}} \qquad . \quad (4\ 5\ 33)$$

$$L_5 = \frac{\Delta_2}{\Delta_2 + 1} \qquad (4\ 5\ 34)$$

$$L_6 = K_1 \, U^{\frac{K_2 - (\Delta_2 + 1)}{(\Delta_2 + 1)}} \qquad . \quad (4.5.35)$$

Once the solution in terms of U is found ϕ can be obtained by backtransformation given by eqn.(4 5 21) and recharge to water table will be given by

$$W = - \frac{\Delta_1}{\Delta_2 + 1} \frac{\partial U}{\partial Z} + U^{\frac{K_2}{\Delta_2 + 1}} \Bigg|_{\text{at } Z=1} \qquad (4\ 5\ 36)$$

4 5 4 SOLUTION

To obtain a soluiton of eqn (4 5 29) following substitultion is used

$$V = (1 - U + \psi)\ e^{-\mu Z - \nu T} \qquad .(4.5\ 37)$$

where μ , ν and ψ are selected such that ,

$$\mu = \frac{L_2}{2L_1} \qquad (4.5.38)$$

$$\nu = \mu^2 L_1 - L_3 \qquad . \quad (4.5\ 39)$$

and $$\psi = (- L_3 + L_4)/L_3 \qquad .(4.5.40)$$

U in terms of V will be given by

$$U = 1 - V\, e^{\mu Z + \nu T} \qquad (4\ 5\ 41)$$

The eqn (4 5 29) now becomes

$$\frac{\partial V}{\partial T} = L_1 V'' \qquad (4\ 5\ 42)$$

with initial condition given by

$$V(Z,0) = [1 - (F(z))^{\Delta_2 + 1} + \psi\ ]\ e^{-\mu Z} \qquad (4.5\ 42\ a)$$

and the boundary condition at water table, Z = 1 as

$$V(1,T) = \psi\ e^{-\nu T - \mu} \qquad (4\ 3\ 42\ b)$$

and boundary condition at ground surface, Z = 0 as

$T_{b1} < T < T_{s1}$

$$L_5 \frac{\partial V}{\partial Z} + (\ \mu L_5 - L_6\ )V = [R + L_5 \frac{\partial \psi}{\partial Z} - L_6 \psi - L_6] e^{-\nu T} \qquad .(4.3\ 42\ c)$$

$T_{s1} < T < T_{d1}$

$$V(0,T) = \psi\ e^{-\nu T} \qquad (4\ 3.42\ d)$$

$T_{d1} < T < T_{b1}$

$$L5 \frac{\partial V}{\partial Z} + (\mu L_5 - L_6)V = [-E + L_5 \frac{\partial \psi}{\partial Z} - L_6 \psi - L_6] e^{-\nu T} \qquad (4\ 3\ 42\ e)$$

Note that in this case

$$\frac{\partial \psi}{\partial Z} = 0 \qquad (4.3.43)$$

Thus finally we obtain the equation and initial and boundary condition as

$$\frac{\partial V}{\partial T} = L_1 \frac{\partial^2 V}{\partial Z^2} \qquad ....(4\ 3.44)$$

with initial condition as

$$V(Z,0) = G(Z) \qquad (4\ 3\ 44\ a)$$

and boundary condition at water table, Z = 1 as

$$V(1,T) = G_2(T) \qquad (4\ 3\ 44\ b)$$

and boundary condition at ground surface, Z = 0 as

$T_{b1} < T < T_{s1}$

$$L_5 \frac{\partial V}{\partial Z} + L_7 V = G_{11}(T) \qquad (4\ 3\ 44\ c)$$

$T_{s1} < T < T_{d1}$

$$V(0,T) = G_{12}(T) \qquad (4\ 3\ 44\ d)$$

$T_{d1} < T < T_{b2}$

$$L_5 \frac{\partial V}{\partial Z} + L_7 V = G_{13}(T) \qquad (4\ 3.44\ e)$$

where

$$L_7 = \mu\ L_5 - L_6 \qquad (4\ 3\ 45)$$

$$G(Z) = [1 - (F(Z))^{\Delta_2 + 1} + \psi\ ]\ e^{-\mu Z} \qquad (4\ 5\ 46)$$

$$G_2(T) = \psi\ e^{-\nu T - \mu} \qquad (4.5.47)$$

$$G_{11}(T) = [R + L_5 \frac{\partial \psi}{\partial Z} - L_6 \psi - L_6] e^{-\nu T} \qquad (4\ 5\ 48)$$

$$G_{12}(T) = \psi\ e^{-\nu T} \qquad (4.5\ 49)$$

$$G_{13}(T) = [-E + L_5 \frac{\partial \psi}{\partial Z} - L_6 \psi - L_6] e^{-\nu T} \qquad (4\ 5\ 50)$$

Solution of eqn (4 3 44) during the three phases is obtained as follows

<u>PHASE-I</u> $T_{b1} < T < T_{s1}$

Solution of eqn (4.5 44) subjected to eqn (4 5.44a), eqn (4.5.44 b) and eqn (4 5.44 c) is given by

$$V = \sum_{N=1,2,3}^{\infty} \beta_N \gamma_{NZ} e^{-L_1^2 \alpha_N T} \left[\varepsilon_N + L_1 \alpha_N^2 \tau_{NT} \right]$$

(4 5 51)

where α_N is the root of

$$L_7 \tan \alpha_N = L_5 \alpha_N \qquad (4\ 5\ 52)$$

and

$$\beta_N = - \frac{2}{L_5 L_7} \qquad (4\ 5\ 53)$$

$$\gamma_{NZ} = - L_5 \alpha_N \cos \alpha_N Z + L_7 \sin \alpha_N Z$$

(4 5.54)

$$\varepsilon_N = \int_0^1 G(Z) \gamma_{NZ} \, dZ \qquad (4\ 5\ 55)$$

$$\tau_{NT} = \int_0^T e^{L_1 \alpha_N^2 \lambda} \left[M_3 G_{11}(\lambda) + M_4 G_2(\lambda) \right] d\lambda$$

(4.5 56)

$$M_1 = \left[- L_5 \sin \alpha_N - \frac{L_7}{\alpha_N} (\cos \alpha_N - 1) \right] \qquad (4\ 5\ 57)$$

$$M_2 = \left[- \frac{L_5}{\alpha_N} (\alpha_N \sin \alpha_N + \cos \alpha_N - 1) - \frac{L_7}{\alpha_N^2} (\alpha_N \cos \alpha_N + \sin \alpha_N) \right]$$

(4.5.58)

$$M_3 = \frac{L_7}{(L_7 - L_5)} [M_1 - M_2] \qquad (4\ 5\ 59)$$

$$M_4 = \frac{1}{(L_7 - L_5)} [-L_5 M_1 + L_7 M_2] \qquad (4.5.60)$$

The two integrals ε_N and τ_{NT} are respectively called as initial and boundary condition integral They can be evaluated as follows

Approximating the expression $[1 -(F(z))^{\Delta_2 + 1} + \psi]$ as

$$[1-(F(z))^{\Delta_2+1}+\psi] = \sum_{i=1,3,5}^{m} a_i^2 Z + b_i Z + c_i \qquad (4\ 5\ 61)$$

$$Z_i \le Z \le Z_{i+2}$$

$$i = 1,3,5, \quad m$$

$$Z_1 = 0 \qquad (4.5\ 62)$$

$$Z_{n+2} = 1 \qquad (4.5.63)$$

so that G(Z) becomes

$$G(Z) = \left[\sum_{i=1,3,5}^{m} a_i^2 Z + b_i Z + c_i\right] e^{-\mu Z} \qquad (4\ 5\ 64)$$

$$Z_i \le Z \le Z_{i+2}$$

Using this the initial condition integral becomes

$$\varepsilon_N = \sum_{i=1,3,5}^{m} \int_{Z_i}^{Z_{i+2}} (a_i Z^2 + b_i Z + c_i) e^{-\mu Z} (-L_5 \alpha_N \cos \alpha_N Z + L_7 \sin \alpha_N Z) dZ$$

$$(4.5\ 65)$$

$$= \sum_{i=1,3,5}^{m} \left[- L_5\ \alpha_N\ J + L_7\ I \right] \qquad (4\ 5.66)$$

where

$$I = \int (a_i Z^2 + b_i Z + c_i) e^{-\mu Z} \sin \alpha_N Z \qquad (4.5\ 67)$$

$$= (a_i Z^2 + b_i Z + c_i)\ I_c - (2a_i Z + b_i) \int I_c + 2a_i \iint I_c$$

$$(4\ 5\ 68)$$

$$J = \int (a_i Z^2 + b_i Z + c_i) e^{-\mu Z} \cos \alpha_N Z \qquad (4.5\ 69)$$

$$= (a_i Z^2 + b_i Z + c_i)\ J_c - (2a_i Z + b_i) \int J_c + 2a_i \iint J_c$$

$$(4.5\ 70)$$

$$I_c = \int e^{-\mu Z} \sin \alpha_N Z \, dZ \tag{4 5 71}$$

$$= - \frac{e^{-\mu Z}}{\alpha_N^2 + \mu^2} \left[\alpha_N \cos \alpha_N Z + \mu \sin \alpha_N Z \right] \tag{4 5 72}$$

$$J_c = \int e^{-\mu Z} \cos \alpha_N Z \, dZ \tag{4 5 73}$$

$$= \frac{e^{-\mu Z}}{\alpha_N^2 + \mu^2} \left[\alpha_N \sin \alpha_N Z - \mu \cos \alpha_N Z \right] \tag{4 5 74}$$

$$\int I_c = - \frac{\alpha_N}{\alpha^2 + \mu^2} J_c - \frac{\mu}{\alpha^2 + \mu^2} I_c \, . \tag{4 5 75}$$

$$\int J_c = \frac{\alpha_N}{\alpha^2 + \mu^2} I_c - \frac{\mu}{\alpha^2 + \mu^2} J_c \, . \tag{4 5 76}$$

$$\int\int I_c = - \frac{\alpha_N}{\alpha^2 + \mu^2} \int J_c - \frac{\mu}{\alpha^2 + \mu^2} \int I \, . . \tag{4 5 77}$$

$$\int\int J_c = - \frac{\alpha_N}{\alpha^2 + \mu^2} \int I_c - \frac{\mu}{\alpha^2 + \mu^2} \int J_c \tag{4 5 78}$$

To evaluate first I_c and J_c and their integrals $\int I_c$, $\int J_c$

$\int\int I_c$ and $\int\int J_c$ are evaluated from which I and J can be found at Z_i and Z_{i+2} Then from eqn (4.5 66) ε_N can be found

To evaluate the boundary condition integral rainfall can be written as

$$R(T) = \sum_{j=1,2,3}^{n} a_j T + b_j \qquad (4.5\ 79)$$

$$T_j < T \leq T_{j+1}$$
$$j = 1, 2, 3 \quad n$$

$$T_1 = 0 \qquad (4\ 5\ 80)$$

$$T_{n+1} = T_{s1} \qquad (4\ 5.81)$$

Using this approximation for rainfall the boundary condition integral becomes

$$\tau_{NT} = \int_0^T e^{L_1 \alpha_N^2 \lambda} \left[M_3 G_{11}(\lambda) + M_4 G_2(\lambda) \right] d\lambda \qquad (4\ 5\ 82)$$

$$= \left[M_3 \ I_{b1} + M_4 I_{b2} \right]_0^T \qquad (4\ 5\ 83)$$

where

$$I_{b1} = \int e^{L_1 \alpha_N^2 \lambda} \left[R - L_6 (1+\psi) \right] e^{-\nu\lambda} \, d\lambda \qquad (4\ 5\ 84)$$

$$= \frac{e^{\sigma T}}{\sigma^2} \left[\sigma \left(a_j T + b_j - L_6 (1+\psi) \right) - a_j \right] \qquad (4\ 5\ 85)$$

$$I_{b2} = \frac{\psi}{\sigma} e^{\sigma T - \mu} \qquad (4\ 5\ 86)$$

$$\sigma = L_1 \alpha_N^2 - \nu \qquad (4\ 5.87)$$

Thus by evaluating I_{b1} and I_{b2}, τ_{NT} can be evaluated.

<u>PHASE-II</u> $T_{s1} < T < T_{d1}$

Solution of eqn (4 5 44) subjected to eqn.(4 5 44 a), eqn (4.5 44 b) and eqn (4.5.44 d) is given by

$$V = 2 \sum_{N=1,2,3}^{\infty} e^{-L_1 N^2 \Pi^2 T} \quad \mathrm{Sin}(N\Pi Z) \quad [\, \varepsilon_N + L_1 N\Pi \, \tau_{NT} \,] \tag{4.5 88}$$

where

$$\varepsilon_N = \int_0^1 G(Z) \sin (N\Pi Z) \, dZ \tag{4.5 89}$$

and

$$\tau_{NT} = \int_0^T e^{L_1 N^2 \Pi^2 T} \quad [G_1(T) - (-1)^N \, G_2(T)] dT \tag{4.5 90}$$

Again approximating G(Z) as in eqn (4.5 64) ε_N is given by

$$\varepsilon_N = \sum_{i=1,3,5}^{m} [I]_{Z_i}^{Z_{i+2}} \tag{4 5.91}$$

where I is given by eqn (4.5 66). Boundary condition integral τ_{NT} is given by

$$\tau_{NT} = \frac{\psi}{L_1 N^2 \Pi^2 - \nu} \; [\, e^{(L_1 N^2 \Pi^2 - \nu)T} - (-1)^N \, e^{(L_1 N^2 \Pi^2 - \nu)T - \mu}$$

$$-1 + (-1)^N \, e^{-\mu} \,] \tag{4 5 92}$$

4 5 5 VALIDITY OF RESULTS

Linearisation of the governing partial differential equation introduces error in the predicted moisture content

Consider a problem in which the soil surface is saturated and infiltration is taking place in the soil which is initially at uniform moisture content. As the water will infiltrate,more and

more volume of soil mass will get saturated Unsaturated flow will continue till the entire soil mass gets saturated. For this moisture content profile at various times has been calculated by analytical as well as numerical approach and the results are shown in fig-(4 5 1) The data used for calculation for the soil considered are

$\lambda = 1\ 8$

$P_b = 75$

$\theta_r = \theta_{\omega p} = 0\ 27$

$\theta_s = 0\ 47$

$K_s = 0\ 217 m/d$

$\theta_i = 0\ 37$

The data has been taken from Jensen and Hanks (1967) and Brooks and Corey (1966) for a silty loam From the fig-(4 5 1) it can be seen that the moisture content upto wetting front can be predicted nicely beyond which it is overestimated Linearised solution overestimates moisture content at all times and this overestimation increases as one goes fatther from this point. The remarkable accuracy with which the linearised solution predicts the time to saturation for a particular point can be noticed from the graph The maximum error in the effective saturation $\frac{(\theta - \theta_r)}{(\theta_s - \theta_r)}$ was 53% whereas in the volumetric moisture content it is 14 54 % The recharge rate to the water table was also calculated by both the methods but errors due to linearisation were found to be unacceptably large(500%) This is because nonuniform error in θ will result in large errors in $\frac{\partial \theta}{\partial X}$ and hence in recharge rates

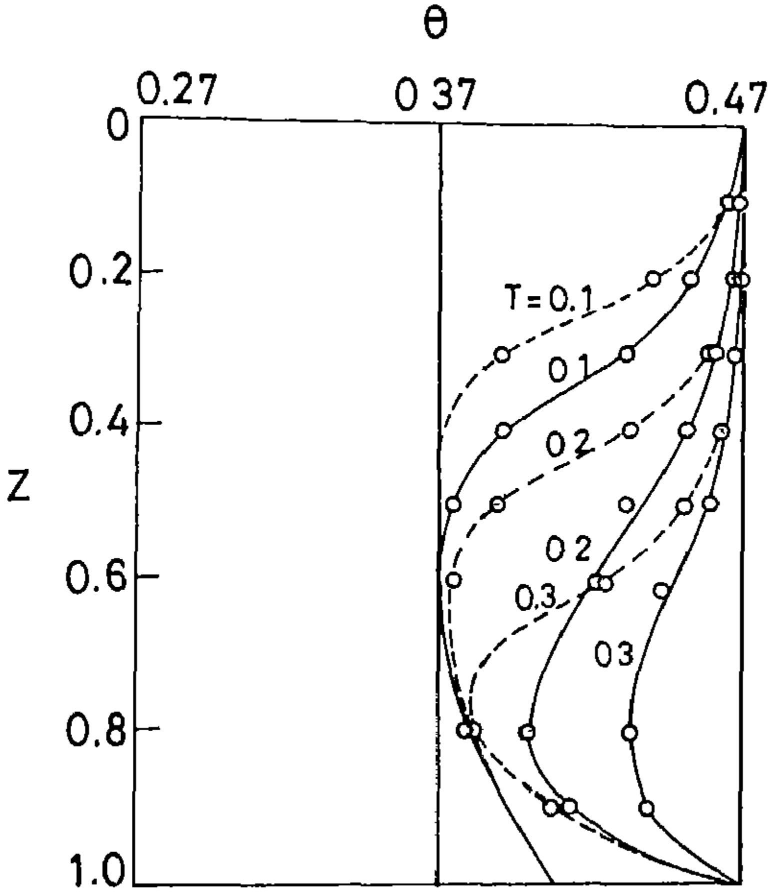


Fig. 4.5.2 Flow in unsaturated zone when surface is saturated.

CHAPTER 5

CONCLUSION

5.1 INTRODUCTION

In this study a general 1-D ground water flow problem in rectanguler coordinates was considered

Exact solution for steady state flow over a horizontal semi-pervious bed was found by using the solutions provided by Sikkema and Van Dam(1982),Nieuwaal and Nijhuis(1979) As exact solution for the case when bed is sloping is difficult to find, a linearised solution was found for this case and effect of linearisation on accuracy of prediction of water table heights and location of ground water divide was studied

For unsteady flow between two parallel drains provided on a horizontal impervious barrier, an exact analytical solution was found and profile of pheratic surface was computed

For general transient flow problem in finite domain exact solution is not available and hence a linearised solutions is found The solution is applicable for arbitary initial condition and transient boundry conditions of any of the three types viz Dirichlet,no flux(Neumann) and Cauchy,for flow over sloping semi-pervious beds Substitutions have been used to remove most of the terms of the linrearised equation and the transformed eqation so obtained is a simple one for which solution can be easily found using the method of separation of variables Subs-tistutions make boundry conditions transient. The solution for transient boundry conditions has been obtained by applying Duhamel's theorem

5.2 GENERAL CONCLUSION

From the above study following conclusions can be made

(1) The linerased solution for flow between two parallel drains over horizontal semi-pervious bed compares well with Young's solution with errors generally less than 5% for all $d/h_o \geq 0.2$. This suggests that linearised analytical solutions can be used effectively

(2) The linearisation approach is valid as long as $d/h_m > 0.7$ as errors will be less than 5% . Instead if 10% errors are allowed the d/hm greater than or equal to 0.6 can be considered

(3) The linearised solution for unsaturated flow predicts the moisture content fairly well till the wave front but thenafter it overestimates the moisture content

(4) The linearised solution for unsaturated flow cannot be used for predicting recharge to water table

5.3 LIMITATIONS OF THE STUDY

The solutions found in the present study are valid only for a homogeneous and isotropic non-deformable porous media Darcy's law has been taken to be valid for both saturated as well as unsaturated flow The artesian head in the underlaying aquifer should not be affected appreciably due to leakage to unconfined aquifer

5.4 SUGGESTIONS FOR FURTHER WORK

(1) The solution based on linearisation approach was found valid for $d_m/h > 0.7$ Efforts may be made to find better results for $d/h_m < 0.7$ also

(2) The solution found here can be used to study the effects of the clogged banks and transient boundry conditions on water table heights

(3) The substiutions used in this study should be tried for obtaining solution for 2-D flow between rectangular boundaries.

(3) Various forms of k-ϕ and p-ϕ relations should be investigated to find the transforming function of the unsaturated flow equation so that a proper form can be selected for accurate prediction of soil moisture content and water table recharge.

(4) Finally the applicability of above approach should be be investigated to the multiple aquifer problems

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